# OPERATIONAL METHODS
# IN
# APPLIED MATHEMATICS


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# PREFACE

FOR some time it has been recognized that by applying the Laplace Transformation

$$f(p) = \int_0^\infty e^{-pt} F(t)\, dt, \quad p > 0,$$

to the differential equations previously treated by the Heaviside Operational Calculus a substitute for these operational methods can be obtained. It is simple and effective. Its principles are easily understood and its technique quickly learned. The difficulties and obscurities of the work of Heaviside and his successors are avoided.

It is the object of this book to describe this new method and to show its use in various branches of applied mathematics.

Chapter I deals with ordinary linear differential equations with constant coefficients. In the next two chapters the methods established in Chapter I are applied to Electric Circuit Theory and Dynamics. These three chapters require no more than the usual knowledge of the Differential and Integral Calculus.

In Chapter IV a more advanced method is given for passing from the Laplace Transform to the function of which it is the transform. This requires some knowledge of the elements of the Theory of Functions of a Complex Variable and the simpler ideas of the Calculus of Residues. It provides a convenient means of verifying that the solutions obtained in the previous chapters, subject to certain assumptions, do, in fact, satisfy all the conditions of their problems.

In Chapter V this method is extended to certain types of partial differential equations. The remaining chapters are devoted to its application in various branches of mathematical physics. These chapters are independent of each other. The reader interested in a particular subject need only study the chapter in which that subject is discussed.

There are numerous examples for solution at the ends of the earlier chapters and a collection for partial differential equations at the close of the book.

We are indebted to Professor E. C. Titchmarsh and Dr. J. H. C. Thompson of Oxford and to Professor V. A. Bailey of Sydney for their helpful criticism of portions of the manuscript.

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H. S. C.
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# CONTENTS

# HISTORICAL INTRODUCTION

1. Heaviside (1850–1925) originally devised his Operational Calculus for the solution of ordinary linear differential equations with constant coefficients and some of the partial differential equations of applied mathematics.

To take a simple example, suppose we have to solve the equation

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_{n-1} \frac{dx}{dt} + a_n x = 1, \quad t > 0, \tag{1}$$

where $a_0$, $a_1, \ldots,$ $a_n$ are constants and

$$x, \quad \frac{dx}{dt}, \quad \ldots, \quad \frac{d^n x}{dt^n} \tag{2}$$

are zero when $t = 0$.

Heaviside replaced $d/dt$ by $p$ and obtained the algebraical equation

$$\phi(p)x = 1,$$

where $$\phi(p) = a_0 p^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n.$$

He regarded $p$ as an 'operator' and his 'operational solution' of the above problem is

$$x = \frac{1}{\phi(p)}, \tag{3}$$

or, more precisely, $$x = \frac{1}{\phi(p)} H(t), \tag{4}$$

where $H(t)$ is a function of the time,† zero when $t < 0$ and unity when $t > 0$.

This 'operational solution', when $\phi(p)$ is a polynomial in $p$, he interpreted by certain rules, of which the most important is the 'Expansion Theorem':

*Let $p_1, p_2, \ldots, p_n$ be the roots of the algebraical equation $\phi(p) = 0$, supposed all different and none of them zero. Then*

$$x = \frac{1}{\phi(0)} + \sum_{r=1}^{n} \frac{e^{p_r t}}{p_r \phi'(p_r)}. \tag{5}$$

† Heaviside wrote **1** for this 'unit function'. His initial conditions are always those of equilibrium.

Another interpretation he obtained by expanding $1/\phi(p)$ in a series of ascending powers of $1/p$. In the above case this will be of the form

$$\frac{b_n}{p^n}+\frac{b_{n+1}}{p^{n+1}}+\dots, \tag{6}$$

and this operates upon $H(t)$.

Heaviside regarded $\frac{1}{p}H(t)$ as equivalent to $\int\limits_0^t H(t)\,dt$, i.e. $t$,

and he found $\frac{1}{p^n}H(t)$ by integrating $n$ times to be $\frac{t^n}{n!}$. (7)

Thus the operational solution $x = \frac{1}{\phi(p)}H(t)$ in (6) gives the actual solution

$$x = b_n\frac{t^n}{n!}+b_{n+1}\frac{t^{n+1}}{(n+1)!}+\dots. \tag{8}$$

Both these rules for solving the equation (1), with its given initial conditions, can be justified, but Heaviside was not much concerned with a rigorous proof. That they gave the correct results seemed to be enough for him.

**2.** However, when he came to deal with partial differential equations, the matter became more obscure. His method can be followed most easily by taking the equation of conduction of heat, $\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}$, which is the same as that for current flow along a cable. Many of his problems deal with this equation.

For example: suppose we have to solve†

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x>0,\ t>0, \tag{1}$$

$$v = 0, \quad \text{when } t = 0,\ x>0, \tag{2}$$

$$v = 1, \quad \text{when } x = 0,\ t>0. \tag{3}$$

Heaviside writes $p$ for $\partial/\partial t$ and replaces these equations by

$$\left.\begin{aligned} &\frac{d^2v}{dx^2}-q^2v = 0, \quad x>0, \quad (q^2 = p/\kappa),\\ \text{and}\quad &v = 1, \quad x = 0.\end{aligned}\right\} \tag{4}$$

† Heaviside, *Electromagnetic Theory* (London, 1899), **2**, 34. This work will be referred to as *E.M.T.*

The 'operational solution', derived from the solution of the ordinary differential equation (4), is

$$v = e^{-qx}H(t) \tag{5}$$

$$= \left(1-qx+\frac{q^2x^2}{2!}-\ldots\right)H(t). \tag{6}$$

Now it is known† that the solution of this problem is

$$v = 1-\frac{2}{\sqrt{\pi}}\int\limits_0^{x/2\sqrt{(\kappa t)}} e^{-\xi^2}\,d\xi \tag{7}$$

$$= 1-\operatorname{erf}\left(\frac{x}{2\sqrt{(\kappa t)}}\right), \quad \text{where} \quad \operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-\xi^2}\,d\xi. \tag{8}$$

Comparing (6) and (8), Heaviside obtained his interpretation of $qH(t)$, $q^2H(t)$, etc.

The simplest, and the one he was most concerned with, is that of $qH(t)$. To obtain this he compared the values of $\partial v/\partial x$ when $x = 0$, obtained from (6) and (7). This gives

$$qH(t) = \frac{1}{\sqrt{(\pi\kappa t)}},$$

i.e. his 'fundamental formula'

$$p^{\frac{1}{2}}H(t) = \frac{1}{\sqrt{(\pi t)}}. \tag{9}$$

From this by differentiation he obtained $p^{\frac{3}{2}}H(t)$ and so on up to $p^{\frac{1}{2}(2n+1)}H(t)$, when $n$ is a positive integer, and

$$p^{\frac{1}{2}(2n+1)}H(t) = (-1)^n\frac{1.3\ldots(2n-1)}{2^n\pi^{\frac{1}{2}}t^{\frac{1}{2}(2n+1)}}. \tag{10}$$

As for $p^nH(t)$, when $n$ is a positive integer, this he took to be zero, since originally he had written $p$ for $d/dt$.

These successive interpretations agree with the equations (6) and (8) above.

† Carslaw, *Conduction of Heat*, 2nd ed. (London, 1921), p. 35. This work will be referred to in future as *C.H.*

For $\frac{1}{p^n}H(t)$ we have already seen that his interpretation was $t^n/n!$, i.e.

$$\frac{t^n}{\Gamma(n+1)}, \quad \text{when } n \text{ is a positive integer.} \tag{11}$$

Now Heaviside regarded $\frac{1}{p^{\frac{1}{2}}}H(t)$ as equivalent to $\frac{1}{p}p^{\frac{1}{2}}H(t)$, so that by the integration of $p^{\frac{1}{2}}H(t)$ he obtained for it $2t^{\frac{1}{2}}/\sqrt{\pi}$, i.e. $t^{\frac{1}{2}}/\Gamma(\frac{3}{2})$.

Integrating again and again he had

$$\frac{1}{p^{\frac{1}{2}(2n+1)}}H(t) = \frac{t^{\frac{1}{2}(2n+1)}}{\Gamma(n+\frac{3}{2})}. \tag{12}$$

It will be noticed that (11) and (12), obtained by different methods, have the same form.

**3.** Having thus given definite meanings to these 'operators', Heaviside considered himself justified in using his method freely.

For instance,† he required the solution of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x > 0, \ t > 0, \tag{1}$$

with $$v = 0, \quad \text{when } t = 0, \ x > 0, \tag{2}$$

and $$\frac{\partial v}{\partial x} = h(v-v_0), \quad \text{when } x = 0, \ t > 0, \tag{3}$$

$h$ being a constant. For these he has the operational equation

$$\left.\begin{aligned} &\frac{d^2v}{dx^2}-q^2v = 0, \quad x > 0, \quad (q^2 = p/\kappa), \\ \text{with} \quad &\frac{dv}{dx} = h(v-v_0), \quad x = 0. \end{aligned}\right\} \tag{4}$$

Thus the operational solution is

$$v = \frac{hv_0}{h+q}e^{-qx},$$

or, more precisely, $$v = \frac{hv_0}{h+q}e^{-qx}H(t). \tag{5}$$

† *E.M.T.* **2**, 14.

First he wanted $v$ for $x = 0$ and $t > 0$. The operational solution is

$$v = \frac{hv_0}{h+q} H(t) \tag{6}$$

$$= \frac{hv_0}{q}\left(1-\frac{h}{q}+\frac{h^2}{q^2}-\dots\right) H(t). \tag{7}$$

Then, from § 2 (12), we have

$$v = \frac{2hv_0\sqrt{(\kappa t)}}{\sqrt{\pi}}\left(1+\frac{2h^2\kappa t}{3}+2^2\frac{h^4(\kappa t)^2}{3\,.\,5}+\dots\right)-v_0(e^{h^2\kappa t}-1). \tag{8}$$

But we can also write (6) in the form

$$v = \frac{v_0}{1+q/h} H(t) = v_0\left(1-\frac{q}{h}+\frac{q^2}{h^2}-\dots\right) H(t). \tag{9}$$

Therefore, from § 2 (10),

$$v = v_0\left(1-\frac{1}{\pi^{\frac{1}{2}}(h^2\kappa t)^{\frac{1}{2}}}+\frac{1}{2\pi^{\frac{1}{2}}(h^2\kappa t)^{\frac{3}{2}}}-\frac{1\,.\,3}{2^2\pi^{\frac{1}{2}}(h^2\kappa t)^{\frac{5}{2}}}-\dots\right). \tag{10}$$

The series (8) is convergent, but (10) is an asymptotic expansion. They are, in fact, the values respectively of

$$v = v_0[1-e^{h^2\kappa t}+e^{h^2\kappa t}\operatorname{erf}\{h\sqrt{(\kappa t)}\}],$$

first as a convergent series and second as an asymptotic expansion, and they are the known results† by orthodox methods for this problem.

The solution for $v$ for any positive $x$ can be similarly obtained.

**4.** One other example from Heaviside's work‡ will be given, as it illustrates his use of the Expansion Theorem when $\phi(p)$ has an infinite number of zeros.

To solve

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad 0 < x < l,\ t > 0, \tag{1}$$

$$v = 0, \quad \text{when } t = 0,\ 0 < x < l, \tag{2}$$

$$\left.\begin{aligned} v &= 1, \quad \text{when } x = 0,\\ v &= 0, \quad \text{when } x = l,\end{aligned}\right\}\quad t > 0. \tag{3}$$

† *C.H.*, p. 52.

‡ *E.M.T.* **2**, 139.

He writes this operationally as

$$\left.\begin{aligned} \frac{d^2v}{dx^2}-q^2v &= 0, \quad 0 < x < l, \\ v &= 1, \quad \text{when } x = 0, \\ v &= 0, \quad \text{when } x = l. \end{aligned}\right\} \tag{4}$$

Thus the operational solution is

$$v = \frac{\sinh q(l-x)}{\sinh ql},$$

and the Expansion Theorem, § 1 (5), gives†

$$v = \frac{l-x}{l}-\frac{2}{\pi}\sum \frac{1}{n}\sin\frac{n\pi x}{l}e^{-\kappa(n^2\pi^2/l^2)t}.$$

Heaviside attached great importance to the Expansion Theorem. As he says in another connexion,‡ 'it goes straight to the final simplified result': and again,|| its 'use, even in comparatively elementary problems, leads to a considerable saving of labour, while in cases involving partial differential equations it is invaluable'. It is significant of his position, however, that, though he does give a sort of discussion of the theorem in the case of systems with a finite number of degrees of freedom, he passes over altogether the question of its applicability in continuous systems.

It is doubtless because of the obscurity, not to say inadequacy, of the mathematical treatment in many of his papers that the importance of his contributions to the theory and practice of the transmission of electric signals by telegraphy and telephony was not recognized in his lifetime and that his real greatness was not then understood.

**5.** Bromwich (1875–1930) was the first to explain, and to a certain extent justify, Heaviside's methods. He made use of the Theory of Functions of a Complex Variable.

† This agrees with the known result: *C.H.*, p. 67.
‡ *E.M.T.* 2, 147.
|| *Electrical Papers*, 2, 373.

He saw that the solution of the problem stated in § 1 is given by

$$x = \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \frac{d\lambda}{\lambda \phi(\lambda)}, \tag{1}$$

the integral being taken along the line $R(\lambda) = \gamma$ in the $\lambda$-plane, $\gamma$ being real and positive and such that all the roots of $\phi(\lambda) = 0$ lie to the left of the line $\lambda = \gamma$.

This integral can be replaced by

$$x = \frac{1}{2i\pi} \int e^{\lambda t} \frac{d\lambda}{\lambda \phi(\lambda)}, \tag{2}$$

where the path is any circle $C$, with its centre at the origin and the zeros of $\phi(\lambda)$ all inside its circumference.

In this form it is easy to show that $x$ satisfies all the conditions of the problem. The Expansion Theorem for this case and the solution in a series of ascending powers of $t$ follow directly from (2).

The interpretation of Heaviside's operational solution $\frac{1}{\phi(p)} H(t)$ is thus

$$\frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \frac{d\lambda}{\lambda \phi(\lambda)},$$

when $\phi(p)$ is a polynomial.

Bromwich extended this work to the case when instead of unity on the right-hand side of the equation we have a function of $t$ (e.g. $ae^{\mu t}$), and also to the case when $x$, $dx/dt, \ldots, d^{n-1}x/dt^{n-1}$ have given arbitrary values when $t = 0$.

He discussed fully the solution of a system of simultaneous equations and showed that the operational method applied there in the most important cases. He found, too, that it could also be used in particular problems for certain partial differential equations, but he was unable to give a general proof for such equations, nor has any yet been given.

His method consisted in finding a solution of the given differential equation and the initial and boundary conditions

in the form of a complex integral over a suitable path. The choice both of the integrand and the contour is not always easy.

His many papers and the exposition of his methods and their development by Jeffreys in his tract, *Operational Methods in Mathematical Physics* (ed. 1, Cambridge, 1927), did much to extend their use. They have since been popular with engineers, and books have been written by engineers giving an exposition of the Operational Calculus supposed to be suitable for their needs. This exposition cannot be said to have been satisfactory. Results which have been established for a particular case are often extended, by analogy, to a more general without justification: and proofs of important theorems are frequently quite inadequate.

After Bromwich, Carson contributed substantially to the theory.† He showed that for the ordinary differential equation treated in § 1 the relation between the expression $1/\phi(p)$ and the function $x(t)$ is given by the equation

$$\frac{1}{\phi(p)} = p\int_0^{\infty} e^{-pt}x(t)\,dt. \tag{3}$$

On this result the treatment given in his important book is based. But, while he established this relation for the particular case referred to, he assumed, quite without proof, that it holds in general.

Van der Pol in 1929 gave a simpler proof of Carson's formula for the ordinary differential equation with constant coefficients and extended it to the case when $x, Dx,..., D^{n-1}x$ take arbitrary values, when $t = 0$. His method is quite elementary and involves multiplying the given equation by $pe^{-pt}$ $(p > 0)$ and integrating with respect to $t$ from 0 to $\infty$.

Before Carson and van der Pol, however, Doetsch had been using the same idea,‡ though he multiplied|| by $e^{-pt}$ instead of

† Cf. Carson, J. R., *Electric Circuit Theory and Operational Calculus* (1926).

‡ See Doetsch, G., *Theorie und Anwendung der Laplace-Transformation* (Berlin, 1937), for the literature.

|| It will be found that, if the multiplier $pe^{-pt}$ is used, the solution of the 'subsidiary equation' obtained by this process is exactly the same in form as the Heaviside operational equation; if the multiplier $e^{-pt}$ is used they always

$pe^{-pt}$. In the language now customary, he applied the Laplace Transformation

$$f(p) = \int_0^\infty e^{-pt} F(t)\, dt, \quad p > 0, \tag{4}$$

to the differential equations of his problems, including the boundary conditions if any. He also made an important change in introducing a new symbol in the 'subsidiary equations', as the operational equations are now frequently called. Heaviside, Bromwich, and their successors used the same symbol, e.g. $x$ in the problem of § 1, both in the differential equation, where it stands for a function of $t$, and in the 'operational equation', where it is a function of $p$.

Finally, he recognized the value of the 'Inversion Theorem' which states that (subject to conditions on $f(p)$ or $F(t)$) the solution of (4) can be obtained in the form

$$F(t) = \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} f(\lambda)\, d\lambda. \tag{5}$$

This point of view has the advantage of bringing together the methods of Bromwich and Carson. Carson's method reduced the solution of the problem to that of the integral equation (3); Bromwich's method consisted virtually in constructing a complex integral solution of type (1) of the original problem. (1) and (3) are connected by the 'Inversion Theorem', so the two methods are complementary.

It is substantially Doetsch's method that is followed in this book, with some modifications when dealing with problems in partial differential equations.

differ by a factor $p$. It was to preserve this correspondence with the Heaviside solution that van der Pol used the multiplier $pe^{-pt}$, but we have preferred, following Doetsch and writers on the Mathematical Theory, to drop the extra factor $p$ which has no mathematical significance and sometimes complicates the algebra.

# CHAPTER I

## ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

**1.** Suppose we are given the equation

$$\frac{d^n x}{dt^n}+a_1\frac{d^{n-1}x}{dt^{n-1}}+\ldots+a_{n-1}\frac{dx}{dt}+a_n x = F(t), \quad t>0, \tag{1}$$

where $a_1$, $a_2$,... are constants, and we require the solution which has

$$x_0,\ x_1,\ldots,\ x_{n-1} \text{ for the values of } x,\ \frac{dx}{dt},\ldots,\ \frac{d^{n-1}x}{dt^{n-1}}, \quad \text{when } t=0. \tag{2}$$

Write as usual $Dx$ for $dx/dt$, $D^2x$ for $d^2x/dt^2$, etc., and let

$$\phi(D) \equiv D^n+a_1 D^{n-1}+\ldots+a_{n-1}D+a_n. \tag{3}$$

When the roots of $\phi(D)=0$ are known and $F(t)$ is such that a particular integral can be found by the usual rules, the customary method of solving the above problem would be to write down the complete solution with its $n$ arbitrary constants and then obtain the values of these constants from the $n$ equations giving $x$, $Dx$,..., $D^{n-1}x$, when $t=0$. We shall now give another and simpler method.

Multiply (1) by $e^{-pt}$, where $p$ is a positive constant, and integrate with respect to $t$ from 0 to $\infty$.

Now

$$\int_0^\infty e^{-pt}Dx\,dt = [e^{-pt}x]_0^\infty+p\int_0^\infty e^{-pt}x\,dt = -x_0+p\int_0^\infty e^{-pt}x\,dt,$$

assuming that $\lim_{t\to\infty}(e^{-pt}x)=0$ and that $\int_0^\infty e^{-pt}x\,dt$ exists when $p$ is greater than some fixed positive number.†

Again,

$$\int_0^\infty e^{-pt}D^2x\,dt = [e^{-pt}Dx]_0^\infty+p\int_0^\infty e^{-pt}Dx\,dt = -x_1+p\int_0^\infty e^{-pt}Dx\,dt,$$

† A similar assumption is made as to $\int_0^\infty e^{-pt}F(t)\,dt$.

assuming that $$\lim_{t\to\infty}(e^{-pt}Dx) = 0.$$

Thus $$\int_0^\infty e^{-pt}D^2x\,dt = -(px_0+x_1)+p^2\int_0^\infty e^{-pt}x\,dt.$$

Proceeding in this way, and with similar assumptions as to $\lim\limits_{t\to\infty}(e^{-pt}D^2x)$, etc., we obtain

$$\int_0^\infty e^{-pt}D^rx\,dt = -(p^{r-1}x_0+p^{r-2}x_1+\ldots+px_{r-2}+x_{r-1})+$$
$$+p^r\int_0^\infty e^{-pt}x\,dt, \quad r \leqslant n. \tag{4}$$

Therefore in place of (1) and (2) we have the equation

$$\left.\begin{aligned}\phi(p)\int_0^\infty e^{-pt}x\,dt = {} & (p^{n-1}x_0+p^{n-2}x_1+\ldots+px_{n-2}+x_{n-1})+\\ & +a_1(p^{n-2}x_0+p^{n-3}x_1+\ldots+px_{n-3}+x_{n-2})+\\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ & +a_{n-2}(px_0+x_1)+\\ & +a_{n-1}x_0+\\ & +\int_0^\infty e^{-pt}F(t)\,dt.\end{aligned}\right\} \tag{5}$$

(5) is called the 'subsidiary equation' corresponding to the given differential equation and initial conditions. It will be noticed that, in forming it, on the left-hand side of (1) we replace $\phi(D)x$ by $\phi(p)\int_0^\infty e^{-pt}x\,dt$, and on the right-hand side

corresponding to $a_{n-1}Dx$ we write $a_{n-1}x_0$,
corresponding to $a_{n-2}D^2x$ we write $a_{n-2}(px_0+x_1)$,
corresponding to $a_{n-3}D^3x$ we write
$$a_{n-3}(p^2x_0+px_1+x_2),$$
. . . . . . . . . . . . . . . . . .

and corresponding to $D^nx$ we write
$$(p^{n-1}x_0+p^{n-2}x_1+\ldots+px_{n-2}+x_{n-1}).$$

To these we add $\int_0^\infty e^{-pt}F(t)\,dt$.

**2.** Our problem is now reduced to finding a value of $x$ which satisfies the 'subsidiary equation'.

As a simple example, consider the equation

$$(D+1)x = 1, \quad t > 0, \tag{1}$$

to be solved with $x = 0$, when $t = 0$.

The subsidiary equation, § 1 (5), is

$$(p+1)\int_0^\infty e^{-pt}x\,dt = \frac{1}{p}.$$

Thus
$$\int_0^\infty e^{-pt}x\,dt = \frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1}. \tag{2}$$

But
$$\frac{1}{p} = \int_0^\infty e^{-pt}\,dt, \qquad p > 0,$$

and
$$\frac{1}{p+1} = \int_0^\infty e^{-(p+1)t}\,dt, \quad p > -1.$$

It follows that (2) is satisfied by $x = 1-e^{-t}$.

In §§ 6, 7, 8 we shall work out a number of examples, $F(t)$ being zero, a constant, $e^{at}$, $\cos at$, $\sin at$, $t^r$, where $r$ is a positive integer, $e^{at}\sin bt$, $e^{at}\cos bt$, $t^re^{at}$, $t^r\cos at$, or $t^r\sin at$. In all these cases, on dividing the subsidiary equation § 1 (5) by $\phi(p)$, we have an equation of the form

$$\int_0^\infty e^{-pt}x\,dt = \frac{g(p)}{h(p)}, \tag{3}$$

where $g(p)$ and $h(p)$ are polynomials in $p$, the degree of the former being at least one less than that of the latter.

We then, by the usual rules, break up $g(p)/h(p)$ into its partial fractions, and by the help of a few known definite integrals, express each of the fractions in the form $\int_0^\infty e^{-pt}u(t)\,dt$. The required solution of (3) can then be written down at once.

Very few definite integrals are, in fact, required; these, and the technique of manipulating them, can be most easily expressed in the Laplace Transformation notation, which we shall now describe.

**3.** We shall frequently write $x(t)$ for $x$ to emphasize its dependence on $t$. We write always

$$\bar{x}(p) = \int_0^\infty e^{-pt}x(t)\,dt,$$

and $\bar{x}(p)$ is called the Laplace Transform of $x(t)$; $p$ is supposed to be a real positive number large enough to make the integral converge.†

TABLE I‡

| $\bar{x}(p) = \int_0^\infty e^{-pt}x(t)\,dt$ | $x(t)$ | |
|---|---|---|
| $\dfrac{1}{p}$ | $1$ | (1) |
| $\dfrac{1}{p^n}$ | $\dfrac{t^{n-1}}{(n-1)!}$, $n$ a positive integer | (2) |
| $\dfrac{1}{p-a}$ | $e^{at}$, $p > \mathbf{R}(a)$ | (3) |
| $\dfrac{a}{p^2+a^2}$ | $\sin at$ | (4) |
| $\dfrac{p}{p^2+a^2}$ | $\cos at$ | (5) |
| $\dfrac{a}{p^2-a^2}$ | $\sinh at$, $p > \lvert a \rvert$ | (6) |
| $\dfrac{p}{p^2-a^2}$ | $\cosh at$, $p > \lvert a \rvert$ | (7) |
| $\dfrac{p}{(p^2+a^2)^2}$ | $\dfrac{t}{2a}\sin at$ | (8) |
| $\dfrac{1}{(p^2+a^2)^2}$ | $\dfrac{1}{2a^3}(\sin at - at\cos at)$ | (9) |

In Table I are collected the only Laplace Transforms which will be needed in Chapters I–III; a fuller table is given in Appendix V.

Of these, (1) to (7) are elementary definite integrals; (8) and (9), which are occasionally of use, are given to illustrate the way in which the table may be extended by differentiation with

† Thus, if $x(t) = e^{at}$, we must have $p > a$. Of course, if there is no such $p$, e.g. if $x(t) = e^{t^2}$, the Laplace Transform does not exist.

‡ The parameter $a$ is real except in (3) when it may be complex.

respect to a parameter. Thus, to derive (8), we take the full statement of (5), namely,

$$\int\limits^{\infty} e^{-pt}\cos at\,dt = \frac{p}{p^2+a^2}.$$

Differentiating both sides with respect to $a$, we obtain

$$\int\limits^{\infty} e^{-pt}t\sin at\,dt = \frac{2ap}{(p^2+a^2)^2},$$

which is equivalent to (8). Similarly, from (4) we obtain (9).

Proceeding in this way† we may obtain the function $x(t)$ whose transform $\bar{x}(p)$ is a function of type $\dfrac{(Ap+B)}{(p^2\pm a^2)^r}$, where $r$ is a positive integer and $A$ and $B$ are constants.

The following simple theorems are of great importance.

Theorem I. *If $\bar{x}_1(p)$ and $\bar{x}_2(p)$ are the transforms of $x_1(t)$ and $x_2(t)$, then $\bar{x}_1(p)\pm\bar{x}_2(p)$ is the transform of $x_1(t)\pm x_2(t)$.*

The proof is obvious.

Theorem II. *If $\bar{x}(p)$ is the transform of $x(t)$ and*

$$\lim_{t\to\infty} e^{-pt}x(t) = 0,$$

*then $p\bar{x}(p)-x(0)$ is the transform of $dx/dt$.*

For

$$\int\limits_0^{\infty} e^{-pt}\frac{dx}{dt}\,dt = \big[e^{-pt}x(t)\big]_0^{\infty}+p\int\limits_0^{\infty} e^{-pt}x(t)\,dt = p\bar{x}(p)-x(0).$$

Theorem III. *If $\bar{x}(p)$ is the transform of $x(t)$ and*

$$\lim_{t\to\infty}\left(e^{-pt}\int\limits_0^t x(\tau)\,d\tau\right) = 0,$$

*then $\dfrac{1}{p}\bar{x}(p)$ is the transform of $\displaystyle\int\limits_0^t x(\tau)\,d\tau$.*

† This method of differentiating with respect to a parameter to find new transforms may be condensed into the following statement:

If $\bar{x}(p, c)$ is the transform of $x(t, c)$, where $c$ is a parameter, and, if $\psi(u)$ is a polynomial in $u$, then

$$\psi\left(\frac{\partial}{\partial c}\right)\bar{x}(p,c) \quad \text{is the transform of} \quad \psi\left(\frac{\partial}{\partial c}\right)x(t,c).$$

For

$$\int_0^\infty e^{-pt}\left[\int_0^t x(\tau)\,d\tau\right]dt$$

$$= -\left[\frac{1}{p}e^{-pt}\int_0^t x(\tau)\,d\tau\right]_0^\infty + \frac{1}{p}\int_0^\infty e^{-pt}\frac{d}{dt}\left[\int_0^t x(\tau)\,d\tau\right]dt$$

$$= \frac{1}{p}\int_0^\infty e^{-pt}x(t)\,dt.$$

As an example of the use of this theorem, on applying it to (4) it follows that, if $\bar{x}(p) = a/p(p^2+a^2)$,

$$x(t) = \int_0^t \sin a\tau\,d\tau = \frac{1}{a}[1-\cos at].$$

THEOREM IV. *If $\bar{x}(p)$ is the transform of $x(t)$, and $p+a > 0$, then $\bar{x}(p+a)$ is the transform of $e^{-at}x(t)$.*

For, if $p+a > 0$, $\int_0^\infty e^{-pt}e^{-at}x(t)\,dt$ exists and is $\bar{x}(p+a)$.

This result is very useful in practice, for it makes it possible to write down the function whose transform is $\dfrac{Ap+B}{(p^2+ap+b)^r}$, where $r$ is a positive integer, by simply completing the square in the denominator. E.g.

(i) $$\bar{x}(p) = \frac{p+8}{p^2+4p+5} = \frac{(p+2)+6}{(p+2)^2+1}.$$

Then, by (4) and (5) and Theorem IV,

$$x(t) = e^{-2t}(\cos t+6\sin t).$$

(ii) $$\bar{x}(p) = \frac{2p+3}{(p^2+4p+8)^2} = \frac{2(p+2)-1}{[(p+2)^2+4]^2}.$$

It follows from (8) and (9) and Theorem IV that

$$x(t) = \tfrac{1}{2}te^{-2t}\sin 2t - \tfrac{1}{16}e^{-2t}(\sin 2t - 2t\cos 2t).$$

(iii) It follows from (2) and Theorem IV that,

$$\text{if } \bar{x}(p) = \frac{1}{(p+a)^n}, \quad \text{then } x(t) = \frac{e^{-at}t^{n-1}}{(n-1)!}, \tag{10}$$

of which (3) is the case $n = 1$.

THEOREM V. *If $\bar{x}(p)$ is the transform of $x(t)$, then $e^{-ap}\bar{x}(p)$, $a > 0$, is the transform of the function $X(t)$, where*

$$\left.\begin{aligned} X(t) &= 0, \qquad 0 < t < a, \\ &= x(t-a), \qquad t > a. \end{aligned}\right\}$$

For

$$\int_0^\infty e^{-pt}X(t)\,dt = \int_a^\infty e^{-pt}x(t-a)\,dt = \int_0^\infty e^{-p(t+a)}x(t)\,dt = e^{-ap}\bar{x}(p).$$

The next two theorems are given here for completeness. Proofs of them are relatively difficult; that of Theorem VI is given in Chapter IV, § 33, and that of Theorem VII in Appendix I.

THEOREM VI. *If $\bar{x}_1(p)$ and $\bar{x}_2(p)$ are the transforms of $x_1(t)$ and $x_2(t)$, then $\bar{x}_1(p)\bar{x}_2(p)$ is the transform of $\int_0^t x_1(\tau)x_2(t-\tau)\,d\tau$, and this is equal to $\int_0^t x_1(t-\tau)x_2(\tau)\,d\tau$.*

Theorem III will be recognized as the case $x_1(t) = 1$ of this result.

THEOREM VII. *If two continuous functions $x_1(t)$ and $x_2(t)$ both have the same Laplace Transform $\bar{x}(p)$, then they are identically equal.*

This is a special case of Lerch's theorem, Appendix I. Its importance is obvious, for it ensures that if, from a known $\bar{x}(p)$ (for example, that which we obtain from the subsidiary equation), we find by any means, e.g. from a Table of Transforms, a continuous function $x(t)$ which has $\bar{x}(p)$ for transform, then $x(t)$ is the unique continuous function with this property.†

**4.** In § 1 we derived from the given differential equation and its initial conditions, and subject to certain assumptions, the Laplace Transform $\bar{x}(p)$ of its solution. If $F(t)$ is one of the simple types of function mentioned in § 2, its Laplace Transform, which appears in the right-hand side of § 1 (5), may be

† It follows also from the full statement of Lerch's theorem (see Appendix I) that, if we find from $\bar{x}(p)$ (e.g. by the use of Theorem V) a function $x(t)$ with only ordinary discontinuities, which has $\bar{x}(p)$ for transform, then $x(t)$ is the only function of this type with $\bar{x}(p)$ for transform.

written down from Table I and we obtain $\bar{x}(p)$ in the form of a quotient $f(p)/g(p)$ of polynomials, in which the degree of the numerator is at least one less than that of the denominator.

To find† $x(t)$ in such cases we need only break up $\bar{x}(p)$ into its partial fractions and then write down from Table I the functions of which the partial fractions are the Laplace Transforms.

If $g(p)$ is of degree $n$ and has zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$, *all different*, this may be done by using the formula‡

$$\frac{f(p)}{g(p)} = \sum_{r=1}^{n} \frac{f(\alpha_r)}{(p-\alpha_r)g'(\alpha_r)}. \tag{1}$$

Then we obtain from § 3 (3)

$$x(t) = \sum_{r=1}^{n} \frac{f(\alpha_r)}{g'(\alpha_r)} e^{\alpha_r t}. \tag{2}$$

If $g(p)$ has some repeated zeros, the ordinary algebraical methods must be used.

It frequently happens that some of the zeros, though all different, are complex. In this case we may either express $\bar{x}(p)$ in partial fractions with linear (complex) denominators, using (1), and subsequently combine conjugate terms in (2) in order to get a real solution, or we may express $\bar{x}(p)$ by the ordinary methods in partial fractions with real quadratic denominators and use § 3 (4)–(7), possibly with Theorem IV.

**5.** In the discussion above it has been assumed that the function $x$ has certain properties, e.g.

$$\lim_{t\to\infty} e^{-pt}x = 0, \quad \lim_{t\to\infty} e^{-pt}Dx = 0, \text{ etc.,} \quad \text{and} \quad \int\limits_0^\infty e^{-pt}x(t)\,dt \text{ exists,}$$

so that, even if we assume Theorem VII, namely, that there is

† In Chapter IV an entirely different method of finding $x(t)$ from $\bar{x}(p)$ is given. This involves the use of an inversion formula and the methods of the Theory of Functions of a Complex Variable. Examples of its use in problems on ordinary differential equations are given in Chapter IV, and all the problems of Chapters I–III may be solved in this way. But, in our view, the method given above is simpler.

‡ Cf. Gibson, *Treatise on the Calculus* (1906), § 120.

only one function $x(t)$ corresponding to $\bar{x}(p)$, there remain gaps in the argument.

It is therefore necessary to verify, without using the assumptions referred to above, that the $x(t)$ we have obtained does satisfy the given differential equation and initial conditions. For differential equations of this type and for a wide range of functions $F(t)$ this verification may be performed using no more than the elementary mathematics of this chapter;† but it is rather tedious, so we postpone the verification to Chapter IV where it is shortened by the use of the methods of the Theory of Functions of a Complex Variable.

**6.** *Examples in which* $F(t) = 0$.

Ex. 1. $$\left.\begin{aligned} &(D^2+3D+2)x = 0, \quad t > 0. \\ &x,\ Dx \text{ equal to } x_0 \text{ and } x_1, \text{ when } t = 0. \end{aligned}\right\}$$

The subsidiary equation is

$$(p^2+3p+2)\bar{x} = (px_0+x_1)+3x_0.$$

Thus $$\bar{x} = \frac{px_0+(x_1+3x_0)}{(p+1)(p+2)} = \frac{2x_0+x_1}{p+1} - \frac{x_0+x_1}{p+2}.$$

Therefore $$x = (2x_0+x_1)e^{-t}-(x_0+x_1)e^{-2t}.$$

Ex. 2. $$\left.\begin{aligned} &D^2(D-1)x = 0, \quad t > 0. \\ &x,\ Dx,\ D^2x \text{ equal to } x_0,\ x_1, \text{ and } x_2, \text{ when } t = 0. \end{aligned}\right\}$$

The subsidiary equation is

$$p^2(p-1)\bar{x} = (p^2x_0+px_1+x_2)-(px_0+x_1).$$

Thus

$$\bar{x} = \frac{p^2x_0+p(x_1-x_0)+x_2-x_1}{p^2(p-1)} = \frac{x_0-x_2}{p}+\frac{x_1-x_2}{p^2}+\frac{x_2}{p-1}.$$

Therefore $$x = (x_0-x_2)+(x_1-x_2)t+x_2 e^t.$$

Ex. 3. $$\left.\begin{aligned} &[(D-a)^2+b^2]x = 0, \quad t > 0. \\ &x,\ Dx \text{ equal to } x_0 \text{ and } x_1, \text{ when } t = 0. \end{aligned}\right\}$$

The subsidiary equation is

$$[(p-a)^2+b^2]\bar{x} = (px_0+x_1)-2ax_0.$$

† See Doetsch, loc. cit., chapter 1.

Thus $$\bar{x} = \frac{(p-a)x_0}{(p-a)^2+b^2}+\frac{x_1-ax_0}{(p-a)^2+b^2}.$$

Therefore $$x = \left[x_0 \cos bt+\frac{x_1-ax_0}{b}\sin bt\right]e^{at}.$$

Ex. 4. $(D^2-2D+2)(D^2+2D-3)x = 0, \quad t > 0,$

i.e. $(D^4-5D^2+10D-6)x = 0.$

$x, Dx, D^2x, D^3x$ equal to 1, 0, 6, and $-14$, when $t = 0$.

The subsidiary equation is

$$(p^2-2p+2)(p^2+2p-3)\bar{x}$$
$$= p^3x_0+p^2x_1+px_2+x_3-5(px_0+x_1)+10x_0$$
$$= p^3+p-4.$$

Thus

$$\bar{x} = \frac{p^3+p-4}{(p^2-2p+2)(p^2+2p-3)} = \frac{p}{p^2-2p+2}-\frac{2}{p^2+2p-3}$$
$$= \frac{(p-1)+1}{(p-1)^2+1}-\frac{2}{(p+1)^2-4}.$$

Therefore $$x = e^t(\cos t+\sin t)-e^{-t}\sinh 2t.$$

**7.** *Examples in which $F(t)$ = constant.*

Ex. 1. $$\left.\begin{array}{l}(D-1)(D-2)(D-3)x = 1, \quad t > 0. \\ x,\ Dx, \text{ and } D^2x \text{ zero, when } t = 0.\end{array}\right\}$$

The subsidiary equation is

$$(p-1)(p-2)(p-3)\bar{x} = \frac{1}{p}.$$

Thus

$$\bar{x} = \frac{1}{p(p-1)(p-2)(p-3)} = -\frac{1}{6p}+\frac{1}{2(p-1)}-\frac{1}{2(p-2)}+\frac{1}{6(p-3)}.$$

Therefore $$x = -\tfrac{1}{6}+\tfrac{1}{2}e^t-\tfrac{1}{2}e^{2t}+\tfrac{1}{6}e^{3t}.$$

Ex. 2. $$\left.\begin{array}{l}[(D-a)^2+b^2]x = 1, \quad t > 0. \\ x,\ Dx \text{ equal to zero, when } t = 0.\end{array}\right\}$$

The subsidiary equation is

$$[(p-a)^2+b^2]\bar{x} = \frac{1}{p}.$$

Thus

$$\bar{x} = \frac{1}{p[(p-a)^2+b^2]} = \frac{1}{a^2+b^2}\left\{\frac{1}{p}-\frac{p-2a}{(p-a)^2+b^2}\right\}$$

$$= \frac{1}{a^2+b^2}\left\{\frac{1}{p}-\frac{p-a}{(p-a)^2+b^2}+\frac{a}{(p-a)^2+b^2}\right\}.$$

Therefore $$x = \frac{1}{a^2+b^2}\left\{1-\left(\cos bt-\frac{a}{b}\sin bt\right)e^{at}\right\}.$$

Ex. 3. $\left.\begin{array}{l} D(D-1)^2x = 4, \quad t > 0. \\ x,\ Dx,\ D^2x \text{ equal to } 1,\ 2, \text{ and } -2, \text{ when } t = 0. \end{array}\right\}$

The subsidiary equation is

$$p(p-1)^2\bar{x} = \frac{4}{p}+(p^2x_0+px_1+x_2)-2(px_0+x_1)+x_0$$

$$= \frac{4}{p}+(p^2+2p-2)-2(p+2)+1$$

$$= \frac{p^3-5p+4}{p}.$$

Thus $$\bar{x} = \frac{p^3-5p+4}{p^2(p-1)^2} = \frac{p^2+p-4}{p^2(p-1)} = \frac{3}{p}+\frac{4}{p^2}-\frac{2}{p-1}.$$

Therefore $$x = 3+4t-2e^t.$$

**8.** *Examples in which $F(t)$ is of the following types: $e^{at}$, $\sin at$ or $\cos at$, $t^r$, $t^re^{at}$, and $t^r\sin at$ or $t^r\cos at$ ($r$ a positive integer).*

Ex. 1. $\left.\begin{array}{l} (D^2-3D+2)x = e^{at}, \quad t > 0. \\ x,\ Dx \text{ are } x_0 \text{ and } x_1, \text{ when } t = 0. \end{array}\right\}\ a \neq 1 \text{ or } 2.$

The subsidiary equation is

$$(p^2-3p+2)\bar{x} = \frac{1}{p-a}+(px_0+x_1)-3x_0.$$

Thus

$$\bar{x} = \frac{1}{(p-a)(p-1)(p-2)}+\frac{px_0+x_1-3x_0}{(p-1)(p-2)}$$

$$= \frac{1}{(a-1)(a-2)(p-a)}+\left(\frac{1}{a-1}+2x_0-x_1\right)\frac{1}{p-1}+$$

$$+\left(-\frac{1}{a-2}+x_1-x_0\right)\frac{1}{p-2}.$$

Therefore

$$x = \frac{1}{(a-1)(a-2)}e^{at}+\left(\frac{1}{a-1}+2x_0-x_1\right)e^{t}+\left(-\frac{1}{a-2}+x_1-x_0\right)e^{2t}.$$

Ex. 2. $\left.\begin{array}{l}(D^2-3D+2)x = e^t, \quad t>0.\\ x,\ Dx \text{ zero, when } t=0.\end{array}\right\}$

The subsidiary equation is

$$(p-1)(p-2)\bar{x} = \frac{1}{p-1}.$$

Thus $$\bar{x} = \frac{1}{(p-1)^2(p-2)} = \frac{1}{p-2}-\frac{1}{p-1}-\frac{1}{(p-1)^2}.$$

Therefore $$x = e^{2t}-(t+1)e^t.$$

Ex. 3. $\left.\begin{array}{l}(D^2+m^2)x = a\cos nt, \quad t>0.\\ x,\ Dx \text{ equal to } x_0 \text{ and } x_1, \text{ when } t=0. \quad n\neq m.\end{array}\right\}$

The subsidiary equation is

$$(p^2+m^2)\bar{x} = a\int\limits_0^\infty e^{-pt}\cos nt\,dt+(px_0+x_1) = \frac{ap}{p^2+n^2}+px_0+x_1.$$

Thus

$$\bar{x} = \frac{a}{m^2-n^2}\left(\frac{p}{p^2+n^2}-\frac{p}{p^2+m^2}\right)+\frac{px_0}{p^2+m^2}+\frac{x_1}{p^2+m^2}.$$

Therefore

$$x = \frac{a}{m^2-n^2}(\cos nt-\cos mt)+x_0\cos mt+\frac{x_1}{m}\sin mt.$$

Ex. 4. $\left.\begin{array}{l}(D^2+n^2)x = a\sin nt, \quad t>0.\\ x,\ Dx \text{ equal to } x_0 \text{ and } x_1, \text{ when } t=0.\end{array}\right\}$

The subsidiary equation is

$$(p^2+n^2)\bar{x} = a\int\limits_0^\infty e^{-pt}\sin nt\,dt+(px_0+x_1) = \frac{an}{p^2+n^2}+px_0+x_1.$$

Thus $$\bar{x} = \frac{an}{(p^2+n^2)^2}+\frac{px_0}{p^2+n^2}+\frac{x_1}{p^2+n^2}.$$

Therefore

$$x = \frac{a}{2n}\left(\frac{1}{n}\sin nt-t\cos nt\right)+x_0\cos nt+\frac{x_1}{n}\sin nt.$$

Ex. 5. $\left.\begin{array}{l}D(D-1)x = t^2, \quad t>0.\\ x,\ Dx \text{ equal to } 0 \text{ and } 1, \text{ when } t=0.\end{array}\right\}$

The subsidiary equation is

$$p(p-1)\bar{x} = \int_0^\infty e^{-pt}t^2\,dt+1 = \frac{2}{p^3}+1.$$

Thus $$\bar{x} = \frac{p^3+2}{p^4(p-1)} = \frac{3}{p-1}-\frac{2}{p^4}-\frac{2}{p^3}-\frac{2}{p^2}-\frac{3}{p}.$$

Therefore $$x = 3e^t-\tfrac{1}{3}t^3-t^2-2t-3.$$

Ex. 6. $$\left.\begin{array}{l}(D^3+1)x = \tfrac{1}{2}t^2e^t, \quad t>0.\\ x,\ Dx,\ D^2x \text{ zero, when } t=0.\end{array}\right\}$$

The subsidiary equation is

$$(p^3+1)\bar{x} = \tfrac{1}{2}\int_0^\infty e^{-(p-1)t}t^2\,dt = \frac{1}{(p-1)^3}, \quad p>1.$$

Thus

$$\bar{x} = \frac{1}{(p-1)^3(p^3+1)}$$
$$= \frac{1}{2(p-1)^3}-\frac{3}{4(p-1)^2}+\frac{1}{8(p-1)}-\frac{1}{24(p+1)}-\frac{p-2}{3[(p-\frac{1}{2})^2+\frac{3}{4}]}.$$

Therefore

$$x = \tfrac{1}{4}(t^2-3t+\tfrac{3}{2})e^t-\tfrac{1}{24}e^{-t}-\tfrac{1}{3}\{\cos\tfrac{1}{2}\sqrt{3}t-\sqrt{3}\sin\tfrac{1}{2}\sqrt{3}t\}e^{\frac{1}{2}t}.$$

Ex. 7. $$\left.\begin{array}{l}(D^2+1)x = t\cos 2t, \quad t>0.\\ x,\ Dx \text{ zero, when } t=0.\end{array}\right\}$$

The subsidiary equation is

$$(p^2+1)\bar{x} = \int_0^\infty e^{-pt}t\cos 2t\,dt = \frac{1}{p^2+4}-\frac{8}{(p^2+4)^2}.$$

Thus

$$\bar{x} = \frac{1}{(p^2+1)(p^2+4)}-\frac{8}{(p^2+1)(p^2+4)^2}$$
$$= \frac{1}{3}\left(\frac{1}{p^2+1}-\frac{1}{p^2+4}\right)-\frac{8}{9}\left(\frac{1}{p^2+1}-\frac{3}{(p^2+4)^2}-\frac{1}{p^2+4}\right)$$
$$= -\frac{5}{9}\,\frac{1}{p^2+1}+\frac{5}{9}\,\frac{1}{p^2+4}+\frac{8}{3}\,\frac{1}{(p^2+4)^2}.$$

Therefore

$$x = -\tfrac{5}{9}\sin t+\tfrac{5}{18}\sin 2t+\tfrac{1}{3}[\tfrac{1}{2}\sin 2t-t\cos 2t].$$

**9.** The method can also be used for the solution of simultaneous ordinary differential equations with constant coefficients. Suppose, for example, that we have to solve the system of $n$ second-order equations

$$\sum_{s=1}^{n} e_{rs} x_s = F_r(t), \quad r = 1, 2,\ldots, n,$$

where
$$e_{rs} = a_{rs} D^2 + b_{rs} D + c_{rs}$$

and, when $t = 0$,

$$x_r = u_r, \quad Dx_r = v_r, \quad r = 1,\ldots, n.$$

Proceeding as in § 1, we find the subsidiary equations

$$\sum_{s=1}^{n} p_{rs} \bar{x}_s = \bar{F}_r(p) + \sum_{s=1}^{n} [(a_{rs} p + b_{rs}) u_s + a_{rs} v_s], \quad r = 1,\ldots, n,$$

where
$$p_{rs} = a_{rs} p^2 + b_{rs} p + c_{rs}.$$

Solving, we obtain $\bar{x}_1,\ldots, \bar{x}_n$ and then, in the usual way, $x_1(t),\ldots, x_n(t)$.

In Chapter IV we shall verify that the values of $x_1,\ldots, x_n$ thus obtained do, in fact, satisfy the given differential equations and the conditions when $t = 0$, provided that the determinant

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

does not vanish.

The case $A = 0$ will not be discussed here as it is difficult and not of much practical importance. It will be seen that when $A = 0$ a differential equation of lower order can be obtained by linear combination of the equations of the system, and this implies connexions between the initial conditions. Clearly the same difficulty may arise when the equations are not all of the same order. Particular examples, in which the $F_r(t)$ are all zero, may be dealt with as in § 10, Exs. 3 and 5.

**10.** In the examples of this section $x_0$, $x_1$, $y_0$, $y_1$ will be used for the values of $x$, $Dx$, $y$, $Dy$, when $t = 0$.

Ex. 1.
$$\left.\begin{aligned} &(3D+2)x+Dy = 1, \quad t > 0. \\ &Dx+(4D+3)y = 0. \\ &x_0 = y_0 = 0. \end{aligned}\right\}$$

The subsidiary equations are

$$\left.\begin{aligned}(3p+2)\bar{x}+p\bar{y} &= \frac{1}{p}, \\ p\bar{x}+(4p+3)\bar{y} &= 0.\end{aligned}\right\}$$

Therefore

$$\bar{x} = \frac{4p+3}{p(p+1)(11p+6)} = \frac{1}{2p}-\frac{1}{5(p+1)}-\frac{33}{10(11p+6)}.$$

Thus
$$x = \tfrac{1}{2}-\tfrac{1}{5}e^{-t}-\tfrac{3}{10}e^{-6t/11}.$$

Also
$$\bar{y} = \frac{-1}{(11p+6)(p+1)} = \frac{1}{5}\left(\frac{1}{p+1}-\frac{11}{11p+6}\right).$$

Thus
$$y = \tfrac{1}{5}(e^{-t}-e^{-6t/11}).$$

Ex. 2.
$$\left.\begin{aligned}(D-1)x-2y &= t, \quad t > 0. \\ -2x+(D-1)y &= t. \\ x_0 = 2,\ y_0 &= 4.\end{aligned}\right\}$$

The subsidiary equations are

$$(p-1)\bar{x}-2\bar{y} = \frac{1}{p^2}+2,$$

$$-2\bar{x}+(p-1)\bar{y} = \frac{1}{p^2}+4.$$

Adding, we have
$$(p-3)(\bar{x}+\bar{y}) = \frac{2}{p^2}+6,$$

i.e.
$$\bar{x}+\bar{y} = 2\frac{3p^2+1}{p^2(p-3)} = 2\left[\frac{28}{9(p-3)}-\frac{1}{3p^2}-\frac{1}{9p}\right].$$

Therefore
$$x+y = 2[\tfrac{28}{9}e^{3t}-\tfrac{1}{3}t-\tfrac{1}{9}].$$

Also, subtracting, we have

$$(p+1)(\bar{x}-\bar{y}) = -2,$$

i.e.
$$\bar{x}-\bar{y} = -\frac{2}{p+1}.$$

Therefore
$$x-y = -2e^{-t}.$$

Thus
$$\left.\begin{aligned}x &= \tfrac{28}{9}e^{3t}-e^{-t}-\tfrac{1}{3}t-\tfrac{1}{9}, \\ y &= \tfrac{28}{9}e^{3t}+e^{-t}-\tfrac{1}{3}t-\tfrac{1}{9}.\end{aligned}\right\}$$

Ex. 3.
$$\begin{aligned}(2D-1)x+(3D-2)y &= te^{t}, \quad t > 0. \\ (2D+1)x+(3D+2)y &= te^{2t}.\end{aligned}$$

In this case, since $2x+4y = t(e^{2t}-e^t)$, when $t = 0$, we must have $x+2y = 0$, and thus

$$x_0+2y_0 = 0.$$

The subsidiary equations are

$$(2p-1)\bar{x}+(3p-2)\bar{y} = \frac{1}{(p-1)^2}-y_0,$$

$$(2p+1)\bar{x}+(3p+2)\bar{y} = \frac{1}{(p-2)^2}-y_0.$$

Therefore

$$[(2p-1)(3p+2)-(2p+1)(3p-2)]\bar{x} = \frac{3p+2}{(p-1)^2}-\frac{3p-2}{(p-2)^2}-4y_0,$$

i.e. $$2p\bar{x} = \frac{3p}{(p-1)^2} \quad \frac{3p}{(p-2)^2}+2\left[\frac{1}{(p-1)^2}+\frac{1}{(p-2)^2}\right]-4y_0.$$

Therefore

$$\bar{x} = \frac{}{2(p-1)^2}-\frac{}{2(p-2)^2}+$$
$$+\left[\frac{1}{p}+\frac{1}{(p-1)^2}-\frac{1}{p-1}+\frac{1}{4p}+\frac{1}{2(p-2)^2}-\frac{1}{4(p-2)}\right]-\frac{2y_0}{p}$$
$$= \frac{5}{2}\frac{1}{(p-1)^2}-\frac{1}{(p-2)^2}+\left(\frac{5}{4}-2y_0\right)\frac{1}{p}-\frac{1}{(p-1)}-\frac{1}{4(p-2)}.$$

Thus $$x = (\tfrac{5}{2}t-1)e^t-(t+\tfrac{1}{4})e^{2t}+(\tfrac{5}{4}-2y_0).$$

Also $$2p\bar{y} = \frac{2p-1}{(p-2)^2}-\frac{2p+1}{(p-1)^2}+2y_0,$$

i.e. $$\bar{y} = \frac{1}{(p-2)^2}-\frac{1}{(p-1)^2}-\frac{1}{2p(p-2)^2}-\frac{1}{2p(p-1)^2}+\frac{y_0}{p}$$
$$\frac{3}{4(p-2)^2}-\frac{3}{2(p-1)^2}+\frac{1}{8(p-2)}+\frac{1}{2(p-1)}+\frac{1}{p}\left(y_0-\frac{5}{8}\right).$$

Therefore $$y = (\tfrac{3}{4}t+\tfrac{1}{8})e^{2t}+(-\tfrac{3}{2}t+\tfrac{1}{2})e^t+(y_0-\tfrac{5}{8}).$$

Ex. 4. $$\left.\begin{aligned}(D^2-4D)x-(D-1)y &= 1,\\ (D+6)x+(D^2-D)y &= e^{4t}.\end{aligned}\right\}\ t > 0.$$

The subsidiary equations are

$$(p^2-4p)\bar{x}-(p-1)\bar{y} = (px_0+x_1)-4x_0-y_0+\frac{1}{p},$$

$$(p+6)\bar{x}+(p^2-p)\bar{y} = x_0+(py_0+y_1)-y_0+\frac{1}{p-4}.$$

Therefore

$$[p(p^2-4p)+(p+6)]\bar{x}$$
$$= p[px_0+x_1-4x_0-y_0]+py_0+x_0+y_1-y_0+\frac{p-3}{p-4}.$$

Thus

$$\bar{x} = \frac{p^2x_0+p(x_1-4x_0)+x_0+y_1-y_0}{(p+1)(p-2)(p-3)}+\frac{1}{(p+1)(p-2)(p-4)}$$

$$= \frac{6x_0-x_1-y_0+y_1}{12(p+1)}+\frac{3x_0-2x_1+y_0-y_1}{3(p-2)}+\frac{3x_1-2x_0-y_0+y_1}{4(p-3)}+$$
$$+\frac{1}{15(p+1)}-\frac{1}{6(p-2)}+\frac{1}{10(p-4)}.$$

Therefore

$$x = \tfrac{1}{12}(6x_0-x_1-y_0+y_1+\tfrac{4}{5})e^{-t}+\tfrac{1}{3}(3x_0-2x_1+y_0-y_1-\tfrac{1}{2})e^{2t}+$$
$$+\tfrac{1}{4}(3x_1-2x_0-y_0+y_1)e^{3t}+\tfrac{1}{10}e^{4t}.$$

Also

$$[p^2(p-1)(p-4)+(p+6)(p-1)]\bar{y} = p(p-4)[py_0+x_0+y_1-y_0]-$$
$$-(p+6)[px_0+x_1-4x_0-y_0]+p-\frac{p+6}{p}.$$

Thus

$$\bar{y} = \frac{p^3y_0+p^2(y_1-5y_0)+p(-6x_0-x_1+5y_0-4y_1)-6(x_1-4x_0-y_0)}{(p+1)(p-1)(p-2)(p-3)}+$$
$$+\frac{p+2}{p(p+1)(p-1)(p-2)}$$

$$= \frac{30x_0-5x_1-5y_0+5y_1}{24(p+1)}+\frac{1}{4(p-1)}(18x_0-7x_1+7y_0-3y_1)-$$
$$\frac{}{3(p-2)}(12x_0-8x_1+4y_0-4y_1)+$$
$$+\frac{1}{8(p-3)}(6x_0-9x_1+3y_0-3y_1)+$$
$$+\frac{1}{p}-\frac{1}{6(p+1)}-\frac{3}{2(p-1)}+\frac{2}{3(p-2)}.$$

Therefore

$$y = \tfrac{1}{24}(-30x_0+5x_1+5y_0-5y_1-4)e^{-t}+$$
$$+\tfrac{1}{4}(18x_0-7x_1+7y_0-3y_1-6)e^{t}+$$
$$+\tfrac{1}{3}(-12x_0+8x_1-4y_0+4y_1+2)e^{2t}+$$
$$+\tfrac{1}{8}(6x_0-9x_1+3y_0-3y_1)e^{3t}+1.$$

Ex. 5.
$$(D^2+1)x+(D^2-2D)y = 0,$$
$$(D^2+D)x+D^2y = 0.$$

These give
$$(D-1)x+2Dy = 0,$$
and therefore
$$x_1-x_0+2y_1 = 0.$$

The subsidiary equations are

$$\left.\begin{aligned}(p^2+1)\bar{x}+p(p-2)\bar{y} &= (px_0+x_1)+(py_0+y_1)-2y_0,\\ p(p+1)\bar{x}+p^2\bar{y} &= (px_0+x_1)+x_0+(py_0+y_1).\end{aligned}\right\}$$

Therefore

$$p[(p^2+1)-(p+1)(p-2)]\bar{x} = p[p(x_0+y_0)+x_1+y_1-2y_0]-$$
$$-(p-2)[p(x_0+y_0)+x_1+x_0+y_1].$$

Thus
$$p(p+3)\bar{x} = px_0+2(x_0+x_1+y_1),$$
and
$$\bar{x} = \frac{2}{3}\left(\frac{x_0+x_1+y_1}{p}\right)+\frac{x_0-2x_1-2y_1}{3(p+3)}.$$

Therefore
$$x = \tfrac{2}{3}(x_1+x_0+y_1)+\frac{x_0-2x_1-2y_1}{3}e^{-3t}.$$

Also

$$p^2[(p^2+1)-(p^2-p-2)]\bar{y}$$
$$= -p(p+1)[p(x_0+y_0)+x_1+y_1-2y_0]+$$
$$+(p^2+1)[p(x_0+y_0)+x_1+x_0+y_1].$$

Therefore

$$\bar{y} = \frac{x_0+x_1+y_1}{3p^2}+\frac{2x_0-4x_1+9y_0-4y_1}{9p}+\frac{4x_1+4y_1-2x_0}{9(p+3)}.$$

Therefore

$$y = \tfrac{1}{3}(x_0+x_1+y_1)t+\tfrac{1}{9}(2x_0-4x_1+9y_0-4y_1)+$$
$$+\tfrac{1}{9}(4x_1+4y_1-2x_0)e^{-3t}.$$

**11.** In all the problems so far considered the function $F(t)$ has been a simple one for which $\bar{F}(p)$ is a rational function of $p$. If this is not the case, or if $F(t)$ is an arbitrary unknown func-

tion, the solution may be obtained formally as an integral by the use of Theorem VI.

For example, suppose we have to solve

$$(D^2+2D+2)x = F(t), \quad t > 0,$$

with $x$, $Dx$ equal to $x_0$ and $x_1$ when $t = 0$.

The subsidiary equation is

$$(p^2+2p+2)\bar{x} = px_0+(x_1+2x_0)+\bar{F}(p);$$

thus
$$\bar{x} = \frac{px_0+(x_1+2x_0)}{p^2+2p+2}+\frac{\bar{F}(p)}{p^2+2p+2}.$$

To find the function whose transform is the second term we take $\bar{x}_1(p) = \bar{F}(p)$ and $\bar{x}_2(p) = 1/(p^2+2p+2)$ in Theorem VI. Then $x_1(t) = F(t)$, $x_2(t) = e^{-t}\sin t$, and we have finally

$$x = x_0\,e^{-t}\cos t+(x_1+x_0)e^{-t}\sin t+\int\limits_0^t F(\tau)e^{-(t-\tau)}\sin(t-\tau)\,d\tau.$$

If $F(t)$ is a simple function such as those previously discussed, this method may, of course, still be used, but the evaluation of the final definite integral may be awkward and it is usually better to proceed as in §§ 6–8.

**12.** When the equation to be solved is of the type

$$\phi(D)x = 1, \quad t > 0,$$

with $x_0, x_1,\ldots, x_{n-1}$ zero and $\phi(D)$ the polynomial

$$D^n+a_1 D^{n-1}+\ldots+a_{n-1}D+a_n,$$

the subsidiary equation is

$$\bar{x}(p) = \frac{1}{p\phi(p)}, \tag{1}$$

and the solution is obtained by breaking up $1/p\phi(p)$ into its partial fractions. If the zeros of $\phi(p)$ are $\alpha_1, \alpha_2,\ldots, \alpha_n$, all different and none of them zero, we find from § 4 (1)

$$\bar{x}(p) = \frac{1}{p\phi(p)} = \frac{1}{p\phi(0)}+\sum_{r=1}^{n}\frac{1}{\alpha_r(p-\alpha_r)\phi'(\alpha_r)}$$

and thus
$$x(t) = \frac{1}{\phi(0)}+\sum_{r=1}^{n}\frac{1}{\alpha_r\,\phi'(\alpha_r)}e^{\alpha_r t}. \tag{2}$$

This is a special case of *Heaviside's Expansion Theorem.* The method can, of course, be extended to the case in which $x_0, x_1, \ldots, x_{n-1}$ are arbitrary constants; also to the corresponding problems where we have to deal with systems of simultaneous linear differential equations with constant coefficients.

Another type of expansion used by Heaviside, namely, that in ascending powers of $t$, can also be deduced from the subsidiary equation (1). It is obtained, in our notation, by expanding $1/p\phi(p)$ in ascending powers of $1/p$ and then using the integral

$$\frac{1}{p^{n+1}} = \int\limits_0^\infty e^{-pt}\frac{t^n}{n!}\,dt.$$

It will be noted that this form would be of use when $t$ is small and the roots of $\phi(p) = 0$ are not known.

A very simple example will illustrate both types of expansions:

To solve
$$L\frac{dI}{dt}+RI = E_0, \quad t > 0,$$

where $L$, $R$, and $E_0$ are constants and $I = 0$ when $t = 0$.

The subsidiary equation is

$$\bar{I} = \frac{E_0}{p(Lp+R)}.$$

(i) For the Expansion Theorem solution we write

$$\bar{I} = \frac{E_0}{R}\left(\frac{1}{p}-\frac{1}{p+R/L}\right).$$

Therefore
$$I = \frac{E_0}{R}(1-e^{-Rt/L}).$$

(ii) For the series solution we write

$$\bar{I} = \frac{E_0}{Lp^2(1+R/Lp)}$$
$$= \frac{E_0}{Lp^2}\left\{1-\frac{R}{Lp}+\frac{R^2}{L^2p^2}-\ldots\right\}.$$

From this we obtain

$$I = \frac{E_0}{L}\left(t-\frac{R}{L}\frac{t^2}{2!}+\frac{R^2}{L^2}\frac{t^3}{3!}-\ldots\right),$$

which agrees with the result obtained in (i).

## EXAMPLES ON CHAPTER I

In the following examples the initial values of $x$, $Dx$, $D^2x$,... so far as required are taken to be $x_0$, $x_1$, $x_2$,... unless numerical values of these are specified. The answer is given at the end of each example.

1. $(D^2+3D+2)x = 4;\quad x_0 = 2,\ x_1 = 0.$

$[x = 2.$

2. $(D+1)(D+2)x = 1+t+t^2.$

$[x = \frac{3}{2}-t+\frac{1}{2}t^2+(2x_0+x_1-2)e^{-t}-(x_0+x_1-\frac{1}{2})e^{-2t}.$

3. $(D^2+n^2)x = a\sin(mt+\alpha);\quad m \neq n;\quad x_0 = x_1 = 0.$

$\left[x = \frac{a}{n(m^2-n^2)}\{m\cos\alpha\sin nt+n\sin\alpha\cos nt-n\sin(mt+\alpha)\}.\right.$

4. $(D^2+n^2)x = a\sin(nt+\alpha);\quad x_0 = x_1 = 0.$

$[x = a\{\sin nt\cos\alpha-nt\cos(nt+\alpha)\}/2n^2.$

5. $(D^2-m^2)x = ae^{mt}+be^{nt};\quad x_0 = x_1 = 0.$

$[x = (a/2m^2)(mte^{mt}-\sinh mt)+$

$+\frac{b}{2m(m^2-n^2)}\{(m-n)e^{-mt}+(m+n)e^{mt}-2me^{nt}\}.$

6. $(D^2+1)x = \sin t\sin 2t.$

$[x = (x_0-\frac{1}{16})\cos t+\frac{1}{16}\cos 3t+(x_1+\frac{1}{4}t)\sin t.$

7. $(D^2-4)x = e^{2t}.$

$[x = x_0\cosh 2t+(\frac{1}{2}x_1-\frac{1}{8})\sinh 2t+\frac{1}{4}te^{2t}.$

8. $(D^3+1)x = 1;\quad x_0 = x_1 = x_2 = 0.$

$[x = 1-\frac{1}{3}e^{-t}-\frac{2}{3}e^{\frac{1}{2}t}\cos\frac{1}{2}t\sqrt{3}.$

9. $(D^3+1)x = t;\quad x_0 = x_1 = x_2 = 0.$

$[x = t+\frac{1}{3}e^{-t}-\frac{1}{3}e^{\frac{1}{2}t}(\cos\frac{1}{2}t\sqrt{3}+\sqrt{3}\sin\frac{1}{2}t\sqrt{3}).$

10. $(D^3+1)x = \frac{1}{2}t^2;\quad x_0 = x_1 = x_2 = 0.$

$[x = \frac{1}{2}t^2-\frac{1}{3}e^{-t}+\frac{1}{3}e^{\frac{1}{2}t}(\cos\frac{1}{2}t\sqrt{3}-\sqrt{3}\sin\frac{1}{2}t\sqrt{3}).$

11. $(D^3+1)x = 1+t+\frac{1}{2}t^2;\quad x_0 = x_2 = 1,\ x_1 = -1.$

$[x = 1+t+\frac{1}{2}t^2+\frac{2}{3}e^{-t}-\frac{2}{3}e^{\frac{1}{2}t}(\cos\frac{1}{2}t\sqrt{3}+\sqrt{3}\sin\frac{1}{2}t\sqrt{3}).$

12. $(D+1)(D+2)(D+3)x = 1+t+t^2;\quad x_0 = x_1 = x_2 = 0.$

$[x = \frac{35}{54}-\frac{4}{9}t+\frac{1}{6}t^2-e^{-t}+\frac{1}{2}e^{-2t}-\frac{4}{27}e^{-3t}.$

13. $D(D+1)(D+2)(D+3)x = 1.$

$[x = (x_0+\frac{11}{6}x_1+x_2+\frac{1}{6}x_3-\frac{11}{36})+\frac{1}{6}t-(3x_1+\frac{5}{2}x_2+\frac{1}{2}x_3-\frac{1}{2})e^{-t}+$

$+(\frac{3}{2}x_1+2x_2+\frac{1}{2}x_3-\frac{1}{4})e^{-2t}-(\frac{1}{3}x_1+\frac{1}{2}x_2+\frac{1}{6}x_3-\frac{1}{18})e^{-3t}.$

14. $(D^4+4D^3+4D^2)x = 0.$

$[4x = 4x_0-3x_2-x_3+(4x_1+4x_2+x_3)t+(3x_2+x_3)e^{-2t}+(2x_2+x_3)te^{-2t}.$

15. $(D^2+1)^2x = \sin t;\quad x_0 = x_1 = x_2 = x_3 = 0.$

$[x = \frac{1}{8}(3-t^2)\sin t-\frac{3}{8}t\cos t.$

16. $(D^2+1)^2x = t\sin t;\quad x_0 = x_1 = x_2 = x_3 = 0.$

$[x = \frac{1}{24}\{(3t-t^3)\sin t-3t^2\cos t\}.$

17. $(D^2+n^2)^2x = a\sin(nt+\alpha)$; $x_0 = x_1 = x_2 = x_3 = 0$.

$$\left[x = \frac{a}{8n^4}\{3\sin nt\cos\alpha - nt(3\cos nt\cos\alpha - \sin nt\sin\alpha) - n^2t^2\sin(nt+\alpha)\}.\right.$$

18. $(D^2+n^2)^2x = a\sin(mt+\alpha)$; $m \neq n$, $x_0 = x_1 = x_2 = x_3 = 0$.

$$\left[x = \frac{1}{2n^3(m^2-n^2)^2}\{2n^3\sin(mt+\alpha) + m(m^2-3n^2)\cos\alpha\sin nt - \right.$$
$$-2n^3\sin\alpha\cos nt + (m^2-n^2)nt(n\sin\alpha\sin nt - m\cos\alpha\cos nt)\}.$$

19. $(D^2+1)(D^2+4)(D^2+9)x = 1$; $x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$.

$$[x = \tfrac{1}{36} - \tfrac{1}{24}\cos t + \tfrac{1}{60}\cos 2t - \tfrac{1}{360}\cos 3t.$$

20. $(D^2-a^2)(D^2-b^2)(D^2-c^2)x = 1$; $x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = 0$.

$$\left[x = -\frac{1}{a^2b^2c^2} + \frac{1}{a^2(a^2-b^2)(a^2-c^2)}\cosh at + \right.$$
$$+\frac{1}{b^2(b^2-a^2)(b^2-c^2)}\cosh bt + \frac{1}{c^2(c^2-a^2)(c^2-b^2)}\cosh ct.$$

21. $(D^5+1)x = 1$; $x_0 = x_1 = x_2 = x_3 = x_4 = 0$.

$$[x = 1 - \tfrac{1}{5}e^{-t} - \tfrac{2}{5}e^{t\cos\pi/5}\cos(t\sin\tfrac{1}{5}\pi) - \tfrac{2}{5}e^{t\cos 3\pi/5}\cos(t\sin\tfrac{3}{5}\pi).$$

22. $(D^{2n}-1)x = 0$; $x_0 = 1$, $x_1 = x_2 = \dots = x_{2n-1} = 0$.

$$\left[x = \frac{1}{n}\cosh t + \frac{1}{n}\sum_{s=1}^{n-1} e^{t\cos s\pi/n}\cos\left(t\sin\frac{s\pi}{n}\right).\right.$$

## *Simultaneous Equations*

23. $Dx = y$, $Dy = z$, $Dz = x$.

$$[x = \tfrac{1}{3}(x_0+y_0+z_0)e^t -$$
$$-\tfrac{1}{3}e^{-\frac{1}{2}t}\{(-2x_0+y_0+z_0)\cos\tfrac{1}{2}t\sqrt{3} - (y_0-z_0)\sqrt{3}\sin\tfrac{1}{2}t\sqrt{3}\}.$$

24.
$$\left.\begin{aligned}(D^2-3D+2)x+(D-1)y &= 0,\\ -(D-1)x+(D^2-5D+4)y &= 0.\end{aligned}\right\}$$
$x_0 = x_1 = y_1 = 0$, $y_0 = 1$.

$$[x = \tfrac{1}{4}(e^t - e^{3t} + 2te^{3t}), \quad y = \tfrac{1}{4}(5e^t - e^{3t} - 2te^{3t}).$$

25.
$$\left.\begin{aligned}(D^2-8)x+\sqrt{6}\,Dy &= 0,\\ -\sqrt{6}\,Dx+(D^2+2)y &= 0.\end{aligned}\right\}$$
$x_0 = 1$, $x_1 = y_0 = y_1 = 0$.

$$[x = \tfrac{1}{2}(3\cosh 2t - \cos 2t), \quad y = \tfrac{1}{2}\sqrt{6}(\sinh 2t - \sin 2t).$$

26.
$$\left.\begin{aligned}(2D^2-D+9)x-(D^2+D+3)y &= 0,\\ (2D^2+D+7)x-(D^2-D+5)y &= 0.\end{aligned}\right\}$$
$x_0 = x_1 = 1$, $y_0 = y_1 = 0$. [See § 9.]

$$[x = \tfrac{1}{3}(e^t + \sin 2t + 2\cos 2t), \quad y = \tfrac{1}{3}(2e^t - \sin 2t - 2\cos 2t).$$

27. $D^2x+aDy-bx = 0,$
$D^2y-aDx-by = 0.$
$x_0 = y_0 = y_1 = 0,\ x_1 = 1.$

$$\left[x = \frac{1}{\alpha-\beta}(\sin\alpha t-\sin\beta t),\quad y = \frac{1}{\alpha-\beta}(\cos\beta t-\cos\alpha t),\right.$$
where $\alpha$, $\beta$ are $\frac{1}{2}\{a\pm\sqrt{(a^2-4b)}\}$ if $a^2 > 4b$.

Discuss also the cases $a^2 \leqq 4b$.

28. $(D^2-4)x-(D+2)y+(D-2)z = 0,$
$2Dx-(D^2-3)y+(D^2-4)z = 0,$
$(D-2)x-y+(D^2-4)z = 0.$
$x_0 = y_0 = z_0 = 1,\ x_1 = 2,\ y_1 = 3,\ z_1 = 1.$

$[x = -\frac{1}{8}e^{t}+\frac{11}{12}e^{3t}+\frac{5}{24}e^{-3t},\quad y = \frac{1}{8}e^{t}+\frac{11}{12}e^{3t}-\frac{1}{24}e^{-3t},\quad z = \frac{4}{5}e^{2t}+\frac{1}{5}e^{-3t}.$

29. $(D^2-1)x+y+z = 0,$
$x+(D^2-1)y+z = 0,$
$x+y+(D^2-1)z = 0.$

$[x = \frac{1}{3}\{(2x_0-y_0-z_0)\cosh t\sqrt{2}+\frac{1}{2}\sqrt{2}(2x_1-y_1-z_1)\sinh t\sqrt{2}+$
$+(x_0+y_0+z_0)\cos t+(x_1+y_1+z_1)\sin t\}.$

30. $Dx_n = -cx_n+cx_{n-1},\quad n = 1, 2,\ldots,$
$Dx_0 = -cx_0,$
$x_0 = 1,\ x_1,\ x_2,\ldots = 0$ at $t = 0,$

$$\left[x_n = \frac{1}{n!}(ct)^n e^{-ct}.\right.$$

31. $(D^2-4)x-(D+2)y+(D-2)z = \sin 2t,$
$2Dx-(D^2-3)y+(D^2-4)z = 0,$
$(D-2)x-y+(D^2-4)z = 0.$
$x_1 = y_0 = y_1 = z_0 = 0.$

$[x = -\frac{8}{65}\sin 2t+\frac{3}{20}\sinh t+\frac{5}{156}\sinh 3t.$

32. $(D-1)x+\frac{1}{4}y-\frac{1}{2}z-\frac{3}{4}u = 0,$
$\frac{1}{8}x+(D-1)y+\frac{3}{8}z+\frac{1}{2}u = 0,$
$-\frac{1}{2}x-\frac{3}{4}y+(D-1)z+\frac{1}{4}u = 0,$
$\frac{3}{8}x+\frac{1}{2}y+\frac{1}{8}z+(D-1)u = 0.$

$[x = x_0e^t-\frac{1}{4}t(y_0-2z_0-3u_0)e^t.$

## CHAPTER II

# ELECTRIC CIRCUIT THEORY

THROUGHOUT this chapter $I$, $V$, $Q$ will be used for current, E.M.F., and charge respectively, $\bar{I}$, $\bar{V}$, $\bar{Q}$ for their Laplace Transforms, and $\overset{0}{I}$, $\overset{0}{V}$, $\overset{0}{Q}$ for their values at $t = 0$.

**13.** *E.M.F. $V$ applied at $t = 0$ to a circuit consisting of inductance $L$, resistance $R$, and capacity $C$ in series. The initial values of $I$ and $Q$ are to be $\overset{0}{I}$ and $\overset{0}{Q}$ respectively.*†

The current is given by the equation

$$L\frac{dI}{dt}+RI+\frac{Q}{C} = V, \tag{1}$$

where the charge $Q$ on the condenser and the current $I$ are connected by‡

$$\frac{dQ}{dt} = I. \tag{2}$$

Multiplying (1) and (2) by $e^{-pt}$, $p > 0$, and integrating with regard to $t$ from 0 to $\infty$ we obtain in the usual way the subsidiary equations

$$(Lp+R)\bar{I}+\frac{1}{C}\bar{Q} = \bar{V}+L\overset{0}{I}, \tag{3}$$

and
$$p\bar{Q} = \bar{I}+\overset{0}{Q},$$

i.e.
$$\bar{Q} = \frac{\bar{I}}{p}+\frac{\overset{0}{Q}}{p}. \tag{4}$$

Eliminating $\bar{Q}$, we have the fundamental equation

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I} = \bar{V}+L\overset{0}{I}-\frac{\overset{0}{Q}}{Cp}. \tag{5}$$

† Note that in particular cases the system has a differential equation of the first order and only one initial value can be prescribed. E.g. if there is no capacity in the circuit only $\overset{0}{I}$, or if there is no inductance only $\overset{0}{Q}$.

‡ This implies that $I$ is positive when flowing towards the high potential side of the condenser.

**14.** *In this section we consider various types of E.M.F. applied at $t = 0$ to the circuit of § 13, the initial charge and current being in all cases zero.*

Ex. 1. *A constant E.M.F. $E$ applied at $t = 0$ to the circuit of § 13. Initial charge and current zero.*

Here $\overset{0}{I} = \overset{0}{Q} = 0$, and $\bar{V} = E/p$, so § 13 (5) becomes

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I} = \frac{E}{p}.$$

Therefore

$$\bar{I} = \frac{E}{L\left(p^2+\frac{R}{L}p+\frac{1}{LC}\right)} = \frac{E}{L[(p+\mu)^2+n^2]},$$

where $$\mu = \frac{R}{2L}, \qquad n^2 = \frac{1}{LC}-\frac{R^2}{4L^2}. \tag{1}$$

It follows, using Theorem IV, that

$$I = \frac{E}{nL}e^{-\mu t}\sin nt, \quad \text{if } n^2 > 0,$$

$$I = \frac{E}{L}te^{-\mu t}, \qquad \text{if } n^2 = 0,$$

and if $n^2 < 0$, putting $n^2 = -k^2$ we find

$$I = \frac{E}{kL}e^{-\mu t}\sinh kt.$$

Ex. 2. *Alternating E.M.F. $E\sin\omega t$ applied at $t = 0$ to the circuit of § 13. Initial charge and current zero.*

Here $V = E\sin\omega t$, so $\bar{V} = \dfrac{\omega E}{p^2+\omega^2}$, and by § 13 (5)

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I} = \frac{\omega E}{p^2+\omega^2}.$$

Thus $$\bar{I} = \frac{Ep\omega}{(Lp^2+Rp+1/C)(p^2+\omega^2)}. \tag{2}$$

Expressing this in partial fractions with quadratic denominators we find

$$\bar{I} = \frac{E}{Z^2}\left\{\frac{X(p+\mu)-\mu X'}{(p+\mu)^2+n^2}-\frac{Xp-R\omega}{p^2+\omega^2}\right\},$$

where $X = L\omega - 1/C\omega$, $X' = L\omega + 1/C\omega$, $Z^2 = X^2 + R^2$, and $n^2$ and $\mu$ are defined in (1). If $n^2 > 0$ it follows that

$$I = \frac{E}{nZ^2} e^{-\mu t}(nX \cos nt - \mu X' \sin nt) - \frac{E}{Z^2}(X \cos \omega t - R \sin \omega t)$$

$$= \frac{E}{Z} \sin(\omega t - \gamma) - \frac{E}{nZ\sqrt{(LC)}} e^{-\mu t} \sin(nt - \delta), \tag{3}$$

where $\qquad \tan\gamma = X/R, \qquad \tan\delta = nX/\mu X'. \qquad (4)$

Ex. 3. *The problem of Ex. 2; it is required to find the 'steady-state' current only.*

As before, $\bar{I}$ is given by (2). Instead of expressing $\bar{I}$ in partial fractions with quadratic denominators as was done in Ex. 2, it could have been expressed in partial fractions with linear (complex) denominators; the partial fractions are then more easily obtained, but the reduction to the final form (3) is longer. But when only the 'steady-state' current is required it is best to proceed in this way.

The roots of the denominator of (2) are $-\mu \pm in$ and $\pm i\omega$; the roots $-\mu \pm in$ give a contribution to $I$ which dies away like $e^{-\mu t}$ and so may be ignored. Thus we need only determine, by § 4 (1), the partial fractions with denominators $p \pm i\omega$. These are

$$\frac{E}{2i(Li\omega + R + 1/Ci\omega)} \frac{1}{(p - i\omega)} - \frac{E}{2i(-Li\omega + R - 1/Ci\omega)} \frac{1}{p + i\omega},$$

which correspond to current

$$\frac{Ee^{i\omega t}}{2i(Li\omega + R + 1/Ci\omega)} - \frac{Ee^{-i\omega t}}{2i(-Li\omega + R - 1/Ci\omega)} = \frac{E}{Z}\sin(\omega t - \gamma),$$

where $\gamma$ is defined in (4).

The *algebra* of this process is almost identical with that of the usual method of finding steady-state solutions (by assuming all quantities proportional to $e^{i\omega t}$), but the subsidiary equation (2) contains also the transient terms if required.

Ex. 4. *Alternating E.M.F. of the same period and damping applied $t = 0$ to the circuit of* § 13 *(oscillatory case, $n^2 > 0$). The initial charge and current zero. It is required to find the subsequent charge in the condenser.*

As in (1), let $n^2 = \dfrac{1}{LC} - \dfrac{R^2}{4L^2}$, and $\mu = \dfrac{R}{2L}$. Then, including for generality a constant phase angle $\alpha$, we take

$$V = Ee^{-\mu t}\sin(nt+\alpha).$$

Thus, by Theorem IV,

$$\bar{V} = E\frac{(p+\mu)\sin\alpha + n\cos\alpha}{(p+\mu)^2+n^2}.$$

And therefore, by § 13 (4) and (5),

$$\bar{Q} = \frac{\bar{I}}{p} = \frac{E}{L}\frac{(p+\mu)\sin\alpha+n\cos\alpha}{[(p+\mu)^2+n^2]^2}.$$

Thus, using Theorem IV and (8) and (9) of § 3,

$$Q = \frac{E}{L}\left[\frac{1}{2n}\sin\alpha\, e^{-\mu t}t\sin nt + \frac{1}{2n^2}\cos\alpha\, e^{-\mu t}\sin nt - \frac{1}{2n}\cos\alpha\, e^{-\mu t}t\cos nt\right]$$

$$= \frac{E}{2nL}e^{-\mu t}\left[\frac{1}{n}\cos\alpha\sin nt - t\cos(nt+\alpha)\right].$$

Ex. 5. *E.M.F. any function $f(t)$ of the time applied at $t = 0$ to circuit of § 13. Initial charge and current zero.*

In this case by § 13 (5) we find, using the notation (1),

$$\bar{I} = \frac{p}{L[(p+\mu)^2+n^2]}\bar{f}(p). \tag{5}$$

Then in Theorem VI we take $\bar{x}_1(p) = \bar{f}(p)$, so that $x_1(t) = f(t)$, and $\bar{x}_2(p) = p/[L(p+\mu)^2+Ln^2]$, so that, in the case $n^2 > 0$,

$$x_2(t) = e^{-\mu t}(n\cos nt - \mu\sin nt)/(nL).$$

Therefore it follows from Theorem VI that

$$I = \frac{1}{nL}\int\limits_0^t e^{-\mu\tau}(n\cos n\tau - \mu\sin n\tau)f(t-\tau)\,d\tau. \tag{6}$$

Ex. 6. *A battery of E.M.F. E connected to the circuit of § 13 at $t = 0$ and short-circuited at $t = T$. Initial charge and current zero.*

The E.M.F. $V$ is here given by

$$\left.\begin{aligned} V &= E, \quad 0 < t < T, \\ &= 0, \quad t > T; \end{aligned}\right\}$$

thus
$$\bar{V} = E \int e^{-pt}\,dt = \frac{E}{p}(1-e^{-pT}).$$

And so, from § 13 (5),

$$\bar{I} = \frac{E}{L\{p^2+(R/L)p+1/LC\}}(1-e^{-pT}).$$

Hence, using Theorem V and the result of Ex. 1 (i), we have, in the case $n^2 > 0$,

$$\left.\begin{aligned} I &= (E/nL)e^{-\mu t}\sin nt, \quad 0 < t < T, \\ &= (E/nL)e^{-\mu t}\sin nt-(E/nL)e^{-\mu(t-T)}\sin n(t-T), \quad t > T. \end{aligned}\right\}$$

A device which is very useful in problems of this type in which the applied E.M.F. is varied by switching processes is the following: the applied E.M.F. $V$ may be regarded as $f(t)+g(t)$, where

$$f(t) = E, \qquad t > 0,$$

and
$$\left.\begin{aligned} g(t) &= 0, \quad 0 < t < T, \\ &= -E, \quad t > T. \end{aligned}\right\}$$

Then, by Theorem V, $\bar{g} = -\dfrac{Ee^{-pT}}{p}$

and thus $\bar{V} = \bar{f}+\bar{g} = \dfrac{E}{p}(1-e^{-pT})$, as before.

## 15. *Electrical networks.*

A complicated circuit may be regarded as built up of elements of the type† discussed in § 13.

Let $R_k$, $L_k$, $C_k$, $Q_k$, $I_k$ correspond to the $k$th element, let $V_k$ be the potential difference between its terminals, and let $M_{kr}$ be

† For the slightly more general case of inductance $L$ and resistance $R$ in series with a leaky condenser (capacity $C$ and resistance $1/G$ in parallel) equation § 13 (5) is replaced by

$$\left[Lp+R+\frac{1}{G+Cp}\right]\bar{I} = L\overset{0}{I}+\bar{V}-\frac{\phantom{Q}}{G+Cp}.$$

the mutual inductance between it and the $r$th element. Then for the current in this element we have

$$\left[L_k \frac{dI_k}{dt}+R_k I_k+\frac{\overset{0}{Q}_k}{C_k}+\frac{1}{C_k}\int\limits_0^t I_k\,dt\right]+\sum_{r\neq k} M_{kr}\frac{dI_r}{dt}=V_k.$$

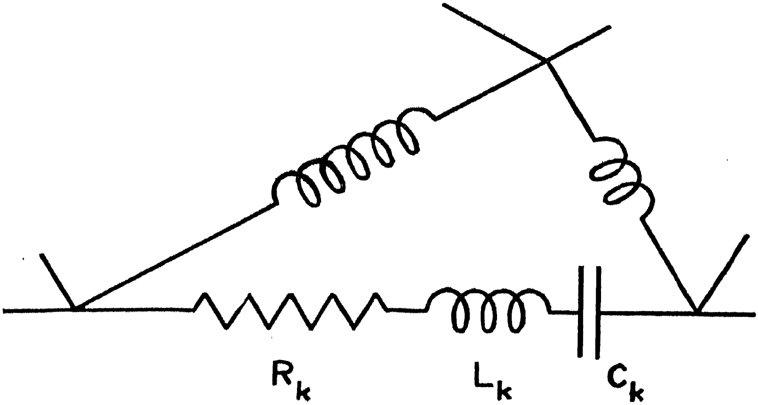


FIG. 1

Proceeding as in § 13, we obtain the subsidiary equation

$$\sum_r z_{kr}\bar{I}_r=\bar{V}_k-\frac{\overset{0}{Q}_k}{pC_k}+L_k\overset{0}{I}_k+\sum_{r\neq k} M_{kr}\overset{0}{I}_r, \tag{1}$$

where

$$z_{kk}=L_k p+R_k+\frac{1}{C_k p},\quad\text{and}\quad z_{kr}=M_{kr}p,\quad\text{if } r\neq k. \tag{2}$$

We shall in future for shortness use $z$ for the 'generalized impedances' (2) and write down subsidiary equations directly in the form (1).

If we add the equations of type (1) for all elements comprising a closed circuit and use Kirchhoff's second law, we obtain

$$\sum\sum_r z_{kr}\bar{I}_r=\sum\left[L_k\overset{0}{I}_k+\sum_{r\neq k} M_{kr}\overset{0}{I}_r-\overset{0}{Q}_k/pC_k\right]+\sum\bar{V}, \tag{3}$$

where the $\sum$ refers to the summation over the members of the closed circuit, and $\sum\bar{V}$ is the sum of the transforms of the *applied* E.M.F.s in the closed circuit.

Kirchhoff's first law, $\sum I = 0$ at a junction, becomes

$$\sum \bar{I} = 0, \quad \text{at a junction.} \tag{4}$$

These equations are sufficient to determine the $\bar{I}_k$ in terms of the initial conditions and the transforms of the applied E.M.F.s. In complicated circuits the denominators of the $\bar{I}_k$ may be of high degree in $p$, so that explicit algebraic solution is clumsy or impossible, but the solution can fairly easily be carried out numerically in actual cases.

**16.** *Examples of simple circuits with non-zero initial currents and charges.*

Ex. 1. *Condenser charged to potential $E$ and discharged at $t = 0$ through an inductive resistance.*

Here $\overset{0}{I} = 0$, $\overset{0}{Q} = CE$, so by § 13 (5)

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I} = -\frac{E}{p}.$$

Thus $$\bar{I} = -\frac{E}{Lp^2+Rp+1/C} = -\frac{E}{L[(p+\mu)^2+n^2]},$$

in the notation of § 14 (1). It follows that

$$I = -\frac{E}{nL}e^{-\mu t}\sin nt, \qquad \text{if } n^2 > 0,$$

$$I = +\frac{E}{L}e^{-\mu t}t, \qquad \text{if } n^2 = 0,$$

$$I = -\frac{E}{kL}e^{-\mu t}\sinh kt, \quad \text{if } n^2 < 0, \text{ where } k^2 = -n^2.$$

To find the charge on the condenser we have by § 13 (4)

$$\bar{Q} = \frac{1}{p}\overset{0}{Q}+\frac{1}{p}\bar{I} = \frac{CE}{p}-\frac{E}{Lp\{p^2+(R/L)p+1/LC\}} = \frac{CE(p+2\mu)}{(p+\mu)^2+n^2}.$$

Therefore, if $n^2 > 0$,

$$Q = \frac{CE}{n}e^{-\mu t}(\mu\sin nt+n\cos nt).$$

Ex. 2. *Steady current $E/R$ is flowing in the circuit of Fig. 2 with the switch $S$ closed. At $t = 0$ the switch is opened.*

Here $\overset{0}{I} = E/R$, $\overset{0}{Q} = 0$, so from § 13 (5)

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I} = \frac{LE}{R}+\frac{E}{p}.$$

So, in the notation of § 14 (1),

$$\bar{I} = \frac{E}{R}\,\frac{(p+R/L)}{\{p^2+(R/L)p+1/LC\}} = \frac{E}{R}\,\frac{p+2\mu}{(p+\mu)^2+n^2}.$$

Therefore

$$I = \frac{E}{Rn}e^{-\mu t}(n\cos nt+\mu\sin nt), \quad \text{if } n^2 > 0.$$

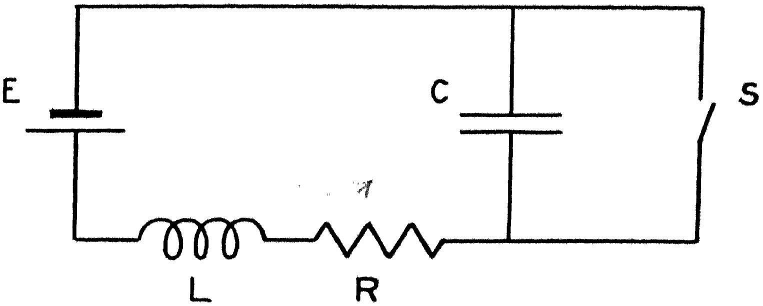


FIG. 2

Ex. 3. *The switch $S$ in the circuit of Fig. 3, which has been closed for time $T$, is opened at $t = 0$.*

It is easy to show that after the battery has been connected for time $T$ (from zero initial conditions) the charge on the condenser is $\overset{0}{Q} = EC$, and the current in the inductance is $\overset{0}{I} = -(E/R)(1-e^{-RT/L})$, measured in the direction of the arrow, i.e. towards the high potential side of the condenser. These are the new initial conditions. Inserting them in § 13 (5), we have

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I} = -\frac{E}{p}-\frac{LE}{R}(1-e^{-RT/L}).$$

Thus

$$\bar{I} = -\frac{E}{L[(p+\mu)^2+n^2]}-\frac{Ep}{R[(p+\mu)^2+n^2]}(1-e^{-RT/L}),$$

in the notation of § 14 (1), and so, if $n^2 > 0$,

$$I = -\frac{E}{nL}e^{-\mu t}\sin nt - \frac{E}{Rn}e^{-\mu t}(n\cos nt - \mu\sin nt)(1-e^{-RT/L}).$$

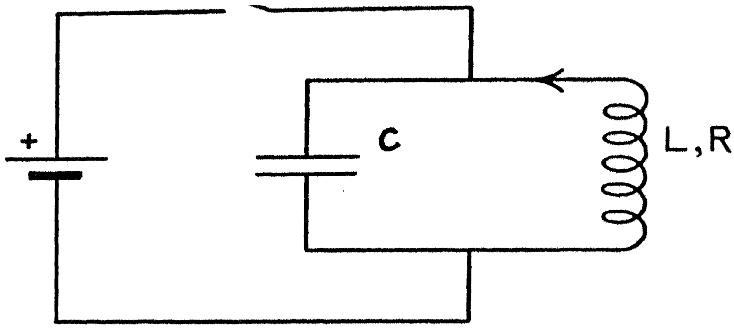


FIG. 3

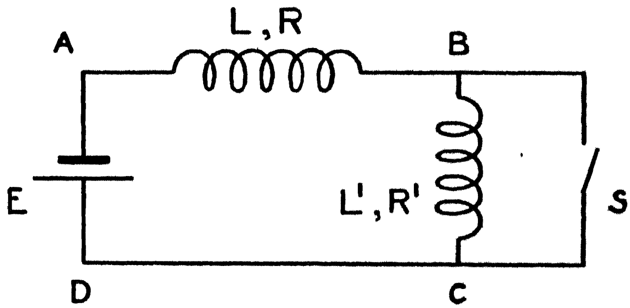


FIG. 4

Ex. 4. *Steady current $E/R$ is flowing in the circuit of Fig. 4 with the switch $S$ closed. At $t = 0$ the switch is opened. To find the subsequent current.*

The initial current is $E/R$ in $L$, and zero in $L'$. Thus, by § 15 (3), for the closed circuit $ABCD$ the subsidiary equation is

$$[(L+L')p+R+R']\bar{I} = \frac{E}{p}+\frac{LE}{R}.$$

Therefore

$$\bar{I} = \frac{LE}{R(L+L')\left[p+\dfrac{R+R'}{L+L'}\right]} + \frac{E}{(L+L')p\left[p+\dfrac{R+R'}{L+L'}\right]},$$

and

$$I = \frac{LE}{R(L+L')}e^{-t(R+R')/(L+L')} + \frac{E}{R+R'}[1-e^{-t(R+R')/(L+L')}]$$

$$= \frac{E}{R+R'} + \frac{E(LR'-L'R)}{R(L+L')(R+R')}e^{-t(R+R')/(L+L')}.$$

Notice that $\lim_{t\to 0} I = \dfrac{E}{R}\dfrac{L}{L+L'}$ and there is an impulsive redistribution† of current between the inductances at $t = 0$.

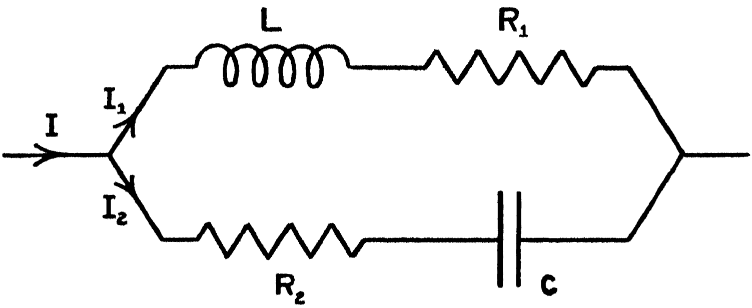


FIG. 5

**17.** *In this section some easy network problems are discussed.*

Ex. 1. *Can the inductive effect of a coil be neutralized by shunting it with a resistance and condenser in series? The initial current and charge are supposed zero.*

Let the currents in the branches be as shown in Fig. 5, and let $V$ be the applied E.M.F. Then the subsidiary equations are

$$(Lp+R_1)\bar{I}_1 = \bar{V} = \left(R_2+\frac{1}{Cp}\right)\bar{I}_2;$$

thus

$$= \bar{I}_1+\bar{I}_2 = \bar{V}\left[\frac{1}{Lp+R_1}+\frac{Cp}{R_2\,Cp+1}\right]$$

$$= \bar{V}\left[\frac{LCp^2+Cp(R_1+R_2)+1}{LCR_2\,p^2+p(L+R_1\,R_2\,C)+R_1}\right].$$

† For further discussion of this problem see Appendix III.

If the system is to behave as a pure resistance, the last bracket must be independent of $p$, so we must have

$$\frac{LC}{LCR_2} = \frac{C(R_1+R_2)}{L+R_1 R_2 C} = \frac{1}{R_1},$$

i.e. $$R_2 = R_1 \quad \text{and} \quad C = \frac{L}{R_1^2},$$

With these values $\bar{I} = \bar{V}/R_1$, so the system behaves as a simple resistance $R_1$, whatever function of the time the applied E.M.F. $V$ may be.

Ex. 2. *Two circuits coupled by mutual induction.*

*Suppose that at $t = 0$ an E.M.F. $V$ is applied in the first circuit, all initial currents and charges being zero.* Then, if

$$z_{11} = L_1 p + R_1 + \frac{1}{C_1 p}, \quad z_{12} = Mp, \quad z_{22} = L_2 p + R_2 + \frac{1}{C_2 p}, \tag{1}$$

the subsidiary equations are

$$z_{11}\bar{I}_1 + z_{12}\bar{I}_2 = \bar{V}, \qquad z_{12}\bar{I}_1 + z_{22}\bar{I}_2 = 0.$$

Solving,†

$$\bar{I}_2 = \frac{z_{12}}{z_{12}^2 - z_{11} z_{22}}\bar{V} \quad \text{and} \quad \bar{I}_1 = -\frac{z_{22}}{z_{12}^2 - z_{11} z_{22}}\bar{V}. \tag{2}$$

In the general case the denominator is a quartic in $p$ and there is no simple algebraic expression for the solution. Special cases are considered in the following examples.

Ex. 3. *The problem of Ex. 2 with $C_1 = C_2 = 0$. A constant E.M.F. $E$ applied to the primary circuit at $t = 0$. It is required to find the secondary current.*

Here $\bar{V} = E/p$, so using (1) and (2)

$$\bar{I}_2 = \frac{ME}{M^2p^2 - (L_1 p + R_1)(L_2 p + R_2)}$$
$$= -\frac{ME}{(L_1 L_2 - M^2)}\frac{1}{(p-\lambda_1)(p-\lambda_2)},$$

† It is known that $L_1 L_2 \geqslant M^2$ and we assume here $L_1 L_2 > M^2$. If $L_1 L_2 = M^2$ (perfect coupling), the determinant of the coefficients of the highest powers of $D$ in the original differential equations vanishes. This is the special case mentioned in § 9 and § 10, Exs. 3 and 5.

where $\lambda_1$ and $\lambda_2$ are the roots (both real and negative since $L_1 L_2 > M^2$) of

$$p^2+\frac{L_1 R_2+R_1 L_2}{L_1 L_2-M^2}p+\frac{R_1 R_2}{(L_1 L_2-M^2)} = 0.$$

So finally, $$I_2 = \frac{ME}{(L_1 L_2-M^2)(\lambda_1-\lambda_2)}[e^{\lambda_1 t}-e^{\lambda_2 t}].$$

Ex. 4. *The problem of Ex. 2 with $V = E\sin\omega t$. It is required to find the 'steady-state' secondary current.*

We have now $$\bar{V} = \frac{E\omega}{p^2+\omega^2},$$

and substituting this in (2) gives

$$\bar{I}_2 = \frac{Mp}{M^2p^2-(L_1 p+R_1+1/C_1 p)(L_2 p+R_2+1/C_2 p)}\,\frac{\omega E}{p^2+\omega^2}.$$

To find the steady-state current we need only evaluate, using § 4 (1), the partial fractions corresponding to the roots $\pm i\omega$ of the denominator.† These give

$$\left\{\frac{R_1 X_2+R_2 X_1}{M\omega}-i\left[M\omega+\frac{R_1 R_2-X_1 X_2}{M\omega}\right]\right\}^{-1}\frac{Ee^{i\omega t}}{2i}+$$

$$+\text{conjugate imaginary} = -\frac{E}{Z_0}\sin(\omega t+\delta),$$

where

$$\tan\delta = \frac{M^2\omega^2+R_1 R_2-X_1 X_2}{R_1 X_2+R_2 X_1},$$

$$Z_0^2 = \left(\frac{R_1 X_2+R_2 X_1}{M\omega}\right)^2+\left[M\omega+\frac{R_1 R_2-X_1 X_2}{M\omega}\right]^2,$$

and $$X_1 = L_1\omega-\frac{1}{C_1\omega}, \qquad X_2 = L_2\omega-\frac{1}{C_2\omega}.$$

Ex. 5. *Two circuits of resistance $R_1$, $R_2$ and inductances $L_1$, $L_2$ are coupled by mutual inductance $M$. At $t = 0$, when a steady current $E/R_1$ is flowing in the primary, this circuit is opened. To find the secondary current.*

† Strictly it should be verified that the other roots of the denominator do give rise to transient terms, i.e. have their real parts negative.

The differential equation is

$$MDI_1+(L_2D+R_2)I_2 = 0,$$

where $I_1 = 0$, $t > 0$, and thus $\bar{I}_1 = 0$, to be solved with $\overset{0}{I}_1 = E/R_1$, $\overset{0}{I}_2 = 0$.

The subsidiary equation is

$$(L_2p+R_2)\bar{I}_2 = M\breve{I}_1 = ME/R_1,$$

and so
$$I_2 = \frac{ME}{L_2R_1}e^{-R_2t/L_2}.$$

Ex. 6. *Two circuits $L_1$, $R_1$ and $L_2$, $R_2$ coupled by mutual inductance $M$. $L_1L_2 = M^2$. A constant E.M.F. $E$ applied at $t = 0$ in the primary circuit. Initial currents zero.*

The differential equations are

$$\left.\begin{aligned}(L_1D+R_1)I_1+MDI_2 &= E,\\ MDI_1+(L_2D+R_2)I_2 &= 0.\end{aligned}\right\}\ t > 0. \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Multiplying (2) by $L_1/M$ and subtracting from (1) we have, using $L_1L_2 = M^2$,

$$R_1I_1-\frac{R_2L_1}{M}I_2 = E, \quad t > 0. \tag{3}$$

At $t = 0$ the E.M.F. in the circuit is zero,† so the initial currents must satisfy

$$R_1\overset{0}{I}_1-\frac{R_2L_1}{M}\overset{0}{I}_2 \quad 0. \tag{4}$$

The values given, $\overset{0}{I}_1 = \overset{0}{I}_2 = 0$, do satisfy (4).

From (1) and (2) the subsidiary equations are

$$\left.\begin{aligned}(L_1p+R_1)\bar{I}_1+Mp\bar{I}_2 &= E/p,\\ Mp\bar{I}_1+(L_2p+R_2)\bar{I}_2 &= 0.\end{aligned}\right\}$$

Solving, we have

$$\bar{I}_1 = \frac{E(L_2p+R_2)}{p[p(L_1R_2+R_1L_2)+R_1R_2]},$$

† The loose statement 'a constant E.M.F. applied at 0' implies that the E.M.F. $V$ in the circuit is

$$\begin{aligned}V &= 0, \quad t = 0,\\ &= E, \quad t > 0.\end{aligned}$$

and $$\bar{I}_2 = -\frac{ME}{p(L_1 R_2 + R_1 L_2) + R_1 R_2}.$$

Thus $$I_1 = \frac{E}{R_1} - \frac{L_1 R_2 E}{R_1(L_1 R_2 + R_1 L_2)} e^{-R_1 R_2 t/(L_1 R_2 + R_1 L_2)}$$

and $$I_2 = -\frac{ME}{L_1 R_2 + R_1 L_2} e^{-R_1 R_2 t/(L_1 R_2 + R_1 L_2)}.$$

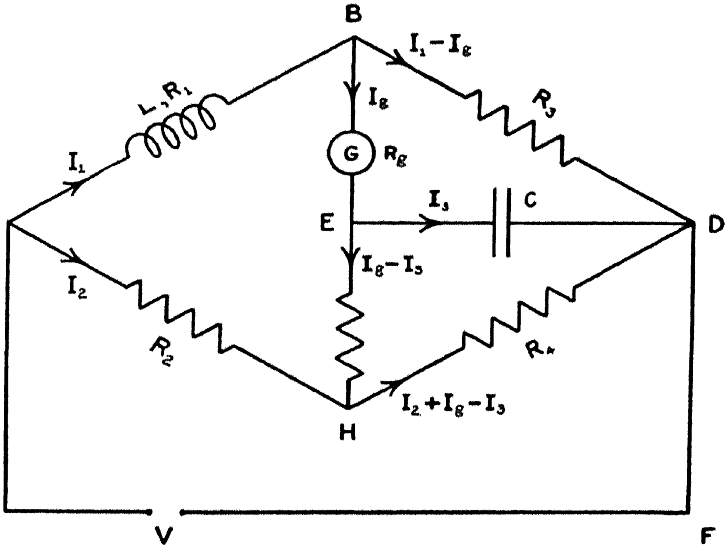


FIG. 6

**18.** *Alternating current bridges.*

Ex. 1. *Anderson's bridge for the comparison of capacity and self-inductance.*

The circuit is shown in Fig. 6. The currents are chosen to satisfy Kirchhoff's first law automatically. An E.M.F. $V$ is applied at $t = 0$ across $AD$ (initial conditions zero) and it is required to find the condition for the galvanometer current $I_g$ to be zero. We write $z_g$ for the impedance of the galvanometer.

Applying Kirchhoff's second law, § 15 (3), to the circuits $ABH$,

$BED$, $DEH$, $ABDF$ in that order gives the subsidiary equations

$$\left.\begin{aligned}(Lp+R_1)\bar{I}_1+z_g\bar{I}_g+R(\bar{I}_g-\bar{I}_3)-R_2\bar{I}_2 &= 0,\\ R_3(\bar{I}_1-\bar{I}_g)-\frac{1}{Cp}\bar{I}_3-z_g\bar{I}_g &= 0,\\ R(\bar{I}_g-\bar{I}_3)+R_4(\bar{I}_2+\bar{I}_g-\bar{I}_3)-\frac{1}{Cp}\bar{I}_3 &= 0,\\ (Lp+R_1)\bar{I}_1+R_3(\bar{I}_1-\bar{I}_g) &= \bar{V}.\end{aligned}\right\}$$

Solving, using determinants, for $\bar{I}_g$ we have (the denominator $\Delta(p)$ is a polynomial in $p$ which we need not calculate)

$$\bar{I}_g = \frac{Cp\bar{V}}{\Delta(p)}\{(R_2R_3-R_1R_4)+p[CR_3(R_2R_4+RR_2+RR_4)-LR_4]\}.$$

Thus, if

$$R_2R_3 = R_1R_4 \quad \text{(the direct current balance condition)}$$

and
$$LR_4 = CR_3(R_2R_4+RR_2+RR_4),$$

$\bar{I}_g = 0$, and so $I_g = 0$ for all $t > 0$, whatever the applied E.M.F. may be, e.g. alternating current of any frequency, or the transient due to switching on a battery.

Ex. 2. *The bridge of Ex. 1 is balanced for direct current, i.e.* $R_2R_3 = R_1R_4$, *and at* $t = 0$ *steady current is flowing from a battery* $E$ *connected to* $AD$. *At* $t = 0$ *the battery circuit is opened. To find the condition that there may be no galvanometer current.*

The initial currents are

$$\overset{0}{I}_1 = \frac{E}{R_1+R_3}, \qquad \overset{0}{I}_2 = \frac{E}{R_2+R_4}, \qquad \overset{0}{I}_3 = \overset{0}{I}_g = 0.$$

The initial charge on the condenser is

$$\overset{0}{Q} = \frac{CER_3}{R_1+R_3}.$$

Also for $t > 0$, $I_2 = -I_1$, thus $\bar{I}_2 = -\bar{I}_1$.

The subsidiary equations for the circuits $ABH$, $BED$, and $DEH$ are

$$\left.\begin{aligned}(Lp+R_1)\bar{I}_1+z_g\bar{I}_g+R(\bar{I}_g-\bar{I}_3)+R_2\bar{I}_1 &= \frac{LE}{R_1+R_3},\\ R_3(\bar{I}_1-\bar{I}_g)-\frac{1}{Cp}\bar{I}_3-z_g\bar{I}_g &= \frac{ER_3}{(R_1+R_3)p},\\ R(\bar{I}_g-\bar{I}_3)-R_4(\bar{I}_1-\bar{I}_g+\bar{I}_3)-\frac{1}{Cp}\bar{I}_3 &= \frac{ER_3}{(R_1+R_3)p}.\end{aligned}\right\}$$

Solving, we have

$$\bar{I}_g = \frac{E(R_3+R_4)}{CpR_4(R_1+R_3)F(p)}[LR_4-CR_3(RR_4+RR_2+R_2R_4)],$$

where, as before, the denominator determinant $F(p)$ need not be evaluated. Thus, if

$$LR_4 = CR_3(RR_4+RR_2+R_2R_4),$$

$$\bar{I}_g = 0,$$

and $$I_g = 0 \quad \text{for all} \quad t>0.$$

Ex. 3. *Rimington's bridge.*

The circuit is shown in Fig. 7. E.M.F. $V$ is supposed applied at $t=0$ across $AK$, all initial currents and charges being zero.

The subsidiary equations for the circuits $ABE$, $BDE$, $DKH$, $ABDF$, respectively, are

$$\left.\begin{aligned}(Lp+R_1)\bar{I}_1+z_g\bar{I}_g-R_2\bar{I}_2 &= 0,\\ R_3(\bar{I}_1-\bar{I}_g)-r_2(\bar{I}_2+\bar{I}_g-\bar{I}_3)-r_1(\bar{I}_2+\bar{I}_g)-z_g\bar{I}_g &= 0,\\ r_2(\bar{I}_2+\bar{I}_g-\bar{I}_3)-\frac{1}{Cp}\bar{I}_3 &= 0,\\ (Lp+R_1)\bar{I}_1+R_3(\bar{I}_1-\bar{I}_g) &= \bar{V}.\end{aligned}\right\}$$

Solving for $\bar{I}_g$, we obtain

$$\bar{I}_g = \frac{\bar{V}}{\Delta(p)}\{LCr_1r_2p^2+p[L(r_1+r_2)+C(R_1r_1r_2-R_2R_3r_2)]+ \\ +R_1(r_1+r_2)-R_2R_3\}, \quad (1)$$

where $\Delta(p)$ is a polynomial in $p$ which we need not calculate. Thus it is impossible† to adjust the resistance so that $\bar{I}_g=0$ for all $\bar{V}$, but there are still two possible ways in which the bridge can be used:

† Unless $r_1=0$. We shall not discuss this case.

(i) As a ballistic bridge. The bridge is first balanced for direct current so that
$$R_1(r_1+r_2) = R_2 R_3. \tag{2}$$

Suppose, then, that a battery of E.M.F. $E$ is switched on at $t = 0$ (the initial currents and charge all being zero) so that

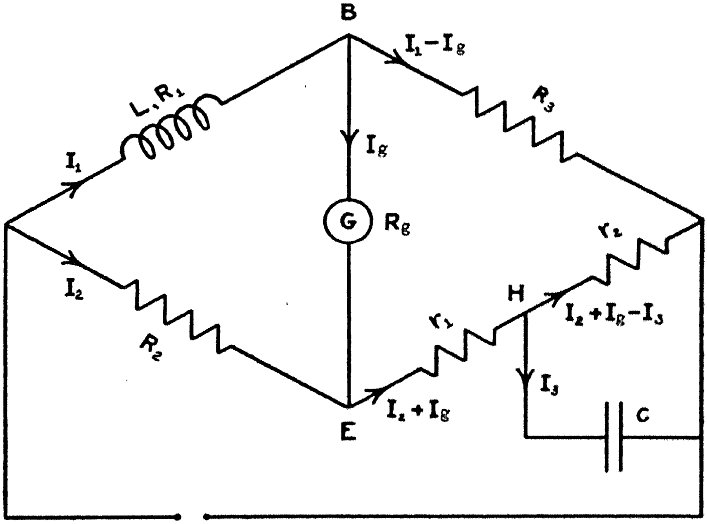


FIG. 7

$\bar{V} = E/p$; we seek the condition that the *total* charge passed through the galvanometer may be zero. This requires†

$$0 = \int_0^\infty I_g\,dt = \lim_{p\to 0}\int_0^\infty e^{-pt}I_g\,dt = \lim_{p\to 0}\bar{I}_g.$$

And, since from (1) and (2),

$$\bar{I}_g = \{E/\Delta(p)\}\{LCr_1 r_2 p+[L(r_1+r_2)+C(r_1 r_2 R_1-R_2 R_3 r_2)]\},$$

we see that $\lim_{p\to 0}\bar{I}_g = 0$ if ‡

$$\frac{L}{C} = \frac{r_2(R_2 R_3-R_1 r_1)}{r_1+r_2} = \frac{R_1 r_2^2}{r_1+r_2}.$$

† The inversion of order of limiting processes below is justifiable since it follows from the form of $I_g$ that $I_g$ consists of a number of exponentially decreasing terms.

‡ It is easy to show that $\Delta(0) \neq 0$.

(ii) *As an alternating-current bridge.* Suppose $V = \sin\omega t$, so that $\bar{V} = \omega/(p^2+\omega^2)$, and that steady current conditions have been attained, then considering only the partial fractions involving the roots $\pm i\omega$ of the denominator of (1), we have for the steady current

$$\{-LCr_1 r_2 \omega^2 + R_1(r_1+r_2) - R_2 R_3 + \\ + i\omega[L(r_1+r_2) + C(R_1 r_1 r_2 - R_2 R_3 r_2)]\} \frac{e^{i\omega t}}{2i\Delta(i\omega)} + \\ + \text{conjugate imaginary.}$$

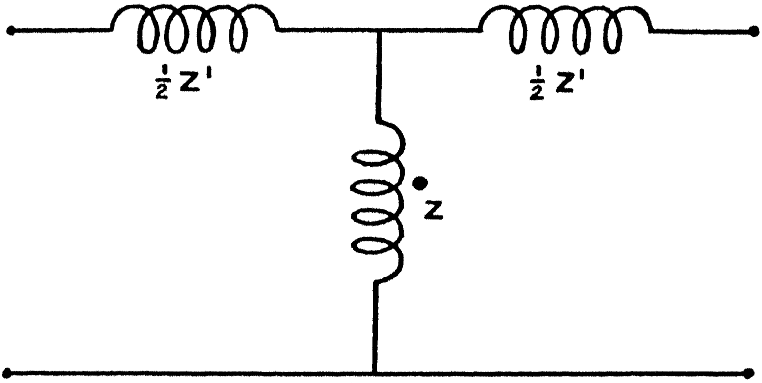


FIG. 8

This will vanish if both the real and imaginary parts of the bracket vanish, i.e. if†

$$LC = \frac{R_1(r_1+r_2)-R_2 R_3}{\omega^2 r_1 r_2} \quad \text{and} \quad \frac{L}{C} = \frac{r_2(R_2 R_3 - R_1 r_1)}{r_1+r_2}.$$

Here, in contrast to Anderson's bridge, there is a balance only for one frequency, and only for the steady state of that frequency.

**19.** *Filter circuits.*

We consider a number $m$ of similar circuit elements ('sections') arranged in tandem so that the output of one is the input of the next. The T section, Fig. 8, is taken as typical. A circuit made up of $m$ such sections, with E.M.F. $V$ applied

† Note that this requires $R_1(r_1+r_2) \neq R_2 R_3$, i.e. the bridge must not be in balance for direct currents.

to the first, and the last closed by impedance $z''$, would appear as in Fig. 9.

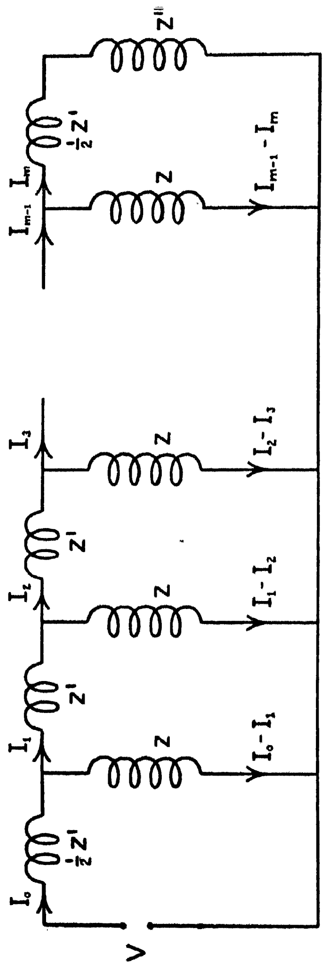


FIG. 9

The currents $I_0, I_1, \ldots$ are chosen as shown in Fig. 9, and the initial currents and charges are supposed zero. Then applying Kirchhoff's second law, § 15 (3), to the meshes successively we obtain the subsidiary equations

$$\left.\begin{aligned} (\tfrac{1}{2}z'+z)\bar{I}_0-z\bar{I}_1 &= \bar{V}, \\ z\bar{I}_0-(2z+z')\bar{I}_1+z\bar{I}_2 &= 0, \\ & \\ z\bar{I}_{m-2}-(2z+z')\bar{I}_{m-1}+z\bar{I}_m &= 0, \\ (z+\tfrac{1}{2}z'+z'')\bar{I}_m-z\bar{I}_{m-1} &= 0. \end{aligned}\right\} \tag{1}$$

Except in the end sections, the transforms of the currents in neighbouring sections are connected by the difference equation

$$z\bar{I}_r-(2z+z')\bar{I}_{r+1}+z\bar{I}_{r+2} = 0. \tag{2}$$

We seek a solution of (2) of type $\bar{I}_r \propto \mu^r$, then substituting in (2), $\mu$ must be a root of the quadratic

$$\mu^2-\left(2+\frac{z'}{z}\right)\mu+1 = 0. \tag{3}$$

Thus, if $\mu_1$ and $\mu_2$ are the roots of (3), a solution of (2) containing two arbitrary constants is

$$\bar{I}_r = A\mu_1^r+B\mu_2^r, \tag{4}$$

and this is in fact the general solution of (2).

The roots of (4) are

$$\left(1+\frac{z'}{2z}\right)\pm\left[\left(1+\frac{z'}{2z}\right)^2-1\right]^{\frac{1}{2}}, \tag{5}$$

which may be written $$e^{\pm\theta}, \tag{6}$$

where $$\cosh\theta = 1+\frac{z'}{2z}. \tag{7}$$

In this notation (4) becomes

$$\bar{I}_r = Ae^{r\theta}+Be^{-r\theta}, \tag{8}$$

where $A$ and $B$ are to be determined in terms of $\bar{V}$ and $z''$ by substituting in the first and last of equations (1), which give

$$\left.\begin{aligned} A(\tfrac{1}{2}z'+z-ze^{\theta})+B(\tfrac{1}{2}z'+z-ze^{-\theta}) &= \bar{V}, \\ Ae^{m\theta}(\tfrac{1}{2}z'+z-ze^{-\theta}+z'')+Be^{-m\theta}(\tfrac{1}{2}z'+z-ze^{\theta}+z'') &= 0. \end{aligned}\right\} \tag{9}$$

Using (7) these become

$$\left.\begin{aligned} -A\sinh\theta+B\sinh\theta &= \bar{V}/z, \\ Ae^{m\theta}(\sinh\theta+z''/z)+Be^{-m\theta}(-\sinh\theta+z''/z) &= 0. \end{aligned}\right\}$$

Solving for $A$ and $B$ and substituting in (8) gives

$$\bar{I}_r = \frac{\bar{V}}{z}\,\frac{\sinh\theta\cosh(m-r)\theta+(z''/z)\sinh(m-r)\theta}{\sinh\theta[\sinh m\theta\sinh\theta+(z''/z)\cosh m\theta]}. \tag{10}$$

As simple cases we consider

Ex. 1.

$$V = E\,(\text{constant}), \quad z'' = 0, \quad z' = R, \quad z = 1/Cp.$$

Here $$\cosh\theta = 1+\tfrac{1}{2}RCp, \tag{11}$$

and (10) becomes

$$\bar{I}_r = \frac{CE\cosh(m-r)\theta}{\sinh\theta\sinh m\theta}. \tag{12}$$

Now†

$$\begin{aligned} \sinh\theta\sinh m\theta &= 2^{m-1}(\cosh\theta-1)(\cosh\theta-\cos\pi/m)\ldots \\ &\qquad\ldots\{\cosh\theta-\cos(m-1)\pi/m\}(\cosh\theta+1) \\ &= 2^{m-1}(\tfrac{1}{2}RCp)(1+\tfrac{1}{2}RCp-\cos\pi/m)\ldots \\ &\qquad\ldots\{1+\tfrac{1}{2}RCp-\cos(m-1)\pi/m\}(2+\tfrac{1}{2}RCp). \end{aligned}$$

† Cf., e.g., Carslaw, *Plane Trigonometry*, 3rd ed. (1930), § 119. In the same way the more general expression (10) may be expressed as a quotient of polynomials in $\cosh\theta$, i.e. as a rational function of $p$.

Thus the zeros of the denominator† of (12) are

$$p = 0, \quad p = -\frac{2}{RC}(1-\cos\pi/m), \quad \ldots,$$

$$p = -\frac{2}{RC}\{1-\cos(m-1)\pi/m\}, \quad p = -\frac{4}{RC},$$

i.e. $$p = -\frac{2}{RC}\left(1-\cos\frac{s\pi}{m}\right), \quad s = 0, 1, 2,\ldots, m.$$

The corresponding values of $\theta$ are $is\pi/m$, $s = 0, 1, 2,\ldots, m$, respectively. Now

$$\frac{d}{dp}(\sinh\theta\sinh m\theta) = \frac{RC}{2}\frac{\cosh\theta\sinh m\theta+m\sinh\theta\cosh m\theta}{\sinh\theta}. \tag{13}$$

Therefore

$$\left[\frac{d}{dp}(\sinh\theta\sinh m\theta)\right]_{p=-\frac{2}{RC}(1-\cos s\pi/m)}$$

$$\begin{aligned} &= \tfrac{1}{2}RCm(-1)^s, \quad && s = 1, 2,\ldots, m-1, \\ &= mRC, && s = 0, \\ &= (-1)^m mRC, && s = m; \end{aligned}$$

the last two results are obtained by taking the limit of (13) as $\theta \to 0$ and $\theta \to i\pi$ respectively. And so, using § 4 (1), we have from (12)

$$I_r = \frac{E}{mR}+\frac{(-1)^r E}{mR}e^{-4t/RC}+\frac{2E}{mR}\sum_{s=1}^{m-1}\cos\frac{rs\pi}{m}e^{-2t(1-\cos s\pi/m)/RC}. \tag{14}$$

Ex. 2.

$$V = E\,(\text{constant}), \quad z'' = 0, \quad z' = Lp+R, \quad z = 1/Cp.$$

Here $$\cosh\theta = 1+\tfrac{1}{2}Cp(Lp+R)$$

and $$\bar{I}_r = \frac{CE\cosh(m-r)\theta}{\sinh\theta\sinh m\theta}. \tag{15}$$

† It may happen that some of these are also roots of the numerator of (12). This need not be allowed for in determining the partial fractions since the corresponding terms (those with $\cos rs\pi/m = 0$) vanish in the final result.

The denominator of (15) vanishes for $\theta = is\pi/m$, $s = 0, 1,\ldots, m$, and the corresponding values of $p$ are the roots of

$$p^2+\frac{R}{L}p+\frac{2}{LC}\left(1-\cos\frac{s\pi}{m}\right) = 0, \quad s = 0, 1,\ldots, m.$$

Thus the zeros of the denominator of (15) are

$$p = 0 \quad \text{and} \quad p = -R/L, \quad \text{corresponding to } s = 0,$$

and

$$p = -\mu \pm i\beta_s, \quad s = 1, 2,\ldots, m,$$

where

$$\mu = R/2L \quad \text{and} \quad \beta_s = \left\{\frac{2}{LC}\left(1-\cos\frac{s\pi}{m}\right)-\frac{R^2}{4L^2}\right\}^{\frac{1}{2}}.$$

Now

$$\frac{d}{dp}(\sinh\theta\sinh m\theta) = \frac{\cosh\theta\sinh m\theta+m\sinh\theta\cosh m\theta}{\sinh\theta}\cdot(LCp+\tfrac{1}{2}RC). \quad (16)$$

Therefore

$$\left[\frac{d}{dp}(\sinh\theta\sinh m\theta)\right]_{p=-\mu\pm i\beta_s} = \pm i(-1)^s mLC\beta_s,$$

$$\text{if } s = 1,\ldots, m-1,$$

$$= \pm 2i(-1)^m mLC\beta_m,$$

$$\text{if } s = m.$$

Also, taking the limit of (16) as $\theta \to 0$,

$$\left[\frac{d}{dp}(\sinh\theta\sinh m\theta)\right]_{p=0} = mRC,$$

$$\left[\frac{d}{dp}(\sinh\theta\sinh m\theta)\right]_{p=-R/L} = -mRC.$$

Using these results we find, on applying § 4 (1) to (15),

$$I_r = \frac{E}{mR}-\frac{E}{mR}e^{-Rt/L}+\frac{(-1)^r E}{mL\beta_m}e^{-\mu t}\sin\beta_m t+$$

$$+\frac{2E}{mL}e^{-\mu t}\sum_{s=1}^{m-1}\frac{\sin\beta_s t}{\beta_s}\cos\frac{rs\pi}{m}.$$

Ex. 3. *Alternating E.M.F. applied to the filter circuit of Fig.* 9.

For definiteness we shall take $V = \sin\omega t$, $z'' = 0$, and consider only the current in the last section, $r = m$. In this case (10) becomes

$$\bar{I}_m = \frac{\omega}{(p^2+\omega^2)z(p)\sinh\theta(p)\sinh m\theta(p)}, \tag{17}$$

where $z(p)$ and $\theta(p)$ are written for $z$ and $\theta$ to emphasize their dependence on $p$.

The denominator of (17) has zeros at $\pm i\omega$ in addition to those of $z(p)\sinh\theta(p)\sinh m\theta(p)$; the latter give rise to terms which may be evaluated as in Exs. 1 and 2; the former give for the part of the current of frequency $\omega/2\pi$

$$I = \frac{e^{i\omega t}}{2iz(i\omega)\sinh\theta(i\omega)\sinh m\theta(i\omega)} + \text{conjugate}. \tag{18}$$

If $\omega$ is such that $\theta(i\omega)$ is complex, say $a+ib$, $\sinh m\theta(i\omega)$ behaves like $e^{m|a|}$ for large $m$ and thus $I$ behaves like $e^{-m|a|}$, i.e. decreases exponentially as the number of sections is increased. Thus the part of the current of frequency $\omega/2\pi$ may be 'stopped', i.e. made as small as we please, by increasing the number of sections.

If $\omega$ is such that $\theta(i\omega)$ is pure imaginary, the hyperbolic functions in (18) become trigonometric functions so that the part of the current of frequency $\omega/2\pi$ does not decrease exponentially as the number of sections is increased. Such frequencies are 'passed' by the filter.

The condition for $\theta(i\omega)$ to be pure imaginary is that $\cosh\theta(i\omega)$ be real and less than 1 in modulus. That is, since $\cosh\theta$ is given by (7), $\frac{1}{2}z'(i\omega)/z(i\omega)$ is to be real and

$$\left|1+\frac{z'(i\omega)}{2z(i\omega)}\right| < 1, \quad \text{i.e. } -1 < \frac{z'(i\omega)}{4z(i\omega)} < 0. \tag{19}$$

As examples consider

(i) *A 'low pass' filter* $z' = Lp$, $z = 1/Cp$. Here (19) requires $-1 < -\frac{1}{4}LC\omega^2 < 0$, i.e. $0 < \omega < 2\omega_1$, where $\omega_1^2 = 1/LC$.

(ii) *A 'band pass' filter.* $z'$ consists of inductance $L_1$ and

capacity $C_1$ in series, $z$ of inductance $L_2$ and capacity $C_2$ in parallel, so that

$$z' = L_1 p + \frac{1}{C_1 p}, \qquad z = \frac{L_2 p}{1+L_2 C_2 p^2}.$$

Then (19) requires

$$-1 < \frac{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}{\omega^2}\frac{L_1 C_2}{4} < 0, \tag{20}$$

where $\omega_1^2 = 1/L_1 C_1$, $\omega_2^2 = 1/L_2 C_2$, and we suppose $\omega_1 < \omega_2$.

Let $\omega_3$ and $\omega_4$, $\omega_3 < \omega_4$, be the roots of

$$L_1 C_2(\omega^2-\omega_1^2)(\omega^2-\omega_2^2) = 4\omega^2,$$

then (20) is satisfied if either $\omega_3 < \omega < \omega_1$ or $\omega_2 < \omega < \omega_4$.

**20.** *The case of an infinite number of sections.*

If $m = \infty$ in § 19, the solution is specially simple since only the root less than unity of § 19 (3) is admissible (i.e. the negative sign in § 19 (5)), and so

$$\bar{I}_r = A\left\{\left(1+\frac{z'}{2z}\right)-\left[\left(1+\frac{z'}{2z}\right)^2-1\right]^{\frac{1}{2}}\right\}^r.$$

Substituting in the first of equations § 19 (1) gives $A$, and thus finally

$$\bar{I}_r = \bar{V}\left[zz'\left(1+\frac{z'}{4z}\right)\right]^{-\frac{1}{2}}\left\{\left(1+\frac{z'}{2z}\right)-\left[\left(1+\frac{z'}{2z}\right)^2-1\right]^{\frac{1}{2}}\right\}^r. \tag{1}$$

For example, consider the simple case

$$z' = R, \quad z = 1/Cp, \quad \bar{V} = E/p.$$

With these values (1) becomes

$$\bar{I}_r = \frac{E}{p}\left[\frac{R}{Cp}(1+\tfrac{1}{4}RCp)\right]^{-\frac{1}{2}}\{(1+\tfrac{1}{2}RCp)-[(1+\tfrac{1}{2}RCp)^2-1]^{\frac{1}{2}}\}^r$$

$$= \frac{2E}{Rk^r}[p(p+2k)]^{-\frac{1}{2}}\{(p+k)-[(p+k)^2-k^2]^{\frac{1}{2}}\}^r,$$

where $k = 2/RC$. Using the formula, Appendix II (35), it follows that

$$I_r = \frac{2E}{R}e^{-kt}I_r(kt),$$

where, on the right, $I_r(kt)$ is the Bessel function of imaginary argument.

**21.** *Any periodic E.M.F. applied to a circuit.*†

Suppose the E.M.F. $V = f(t)$ has period $2T$, so that

$$f(t+2rT) = f(t)$$

for any integer $r$.

Then for the Laplace Transform of $f(t)$ we have

$$\begin{aligned}\bar{f}(p) = \int\limits_0^\infty e^{-pt}f(t)\,dt &= \sum_{r=0}^{\infty} \int\limits_{2rT}^{2(r+1)T} e^{-pt}f(t)\,dt \\ &= (1+e^{-2pT}+e^{-4pT}+\ldots)\int\limits_0^{2T} e^{-pt}f(t)\,dt \\ &= \frac{1}{1-e^{-2pT}}\int\limits_0^{2T} e^{-pt}f(t)\,dt. \qquad (1)\end{aligned}$$

As an illustration of the method of procedure consider the E.M.F.‡

$$\left.\begin{aligned} f(t) &= 1, \quad 2rT < t < (2r+1)T, \\ &= 0, \quad (2r+1)T < t < (2r+2)T, \end{aligned}\right\}$$

applied at $t = 0$ to the circuit of § 13 with

$$R = 0 \quad \text{and} \quad \overset{0}{I} = \overset{0}{Q} = 0.$$

Then $$\int\limits_0^{2T} e^{-pt}f(t)\,dt = \frac{1}{p}(1-e^{-pT}),$$

and thus by (1)

$$\bar{f} = \frac{1-e^{-pT}}{p(1-e^{-2pT})} = \frac{1}{p(1+e^{-pT})}.$$

Hence from § 13 (5), putting $R = 0$ and $n^2 = 1/LC$,

$$\bar{I} = \frac{1}{L(p^2+n^2)(1+e^{-pT})}.$$

Thus, using the Inversion Theorem (§§ 28, 29),

$$I = \frac{1}{2\pi iL}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}\,d\lambda}{(\lambda^2+n^2)(1+e^{-\lambda T})}. \qquad (2)$$

† This problem requires the methods of Chapter IV; it is given here for completeness.

‡ For discussion of other wave forms see McLachlan, *Phil. Mag.* (7), **24** (1937), 1055.

The integrand of (2) is a single-valued function of $\lambda$ with simple poles† at $\lambda = \pm in$, and at

$$\lambda = \pm\frac{(2r+1)\pi i}{T}, \quad r = 0, 1, 2,\dots.$$

The residue at the pole $\lambda = in$ is

$$\frac{e^{int}}{2in(1+e^{-inT})},$$

and that at the pole $\lambda = (2r+1)\pi i/T$ is

$$\frac{e^{(2r+1)\pi it/T}}{T[n^2-(2r+1)^2\pi^2/T^2]}.$$

It can be shown, as in the examples of Chapter IV, by using the contour of Fig. 10 that the line integral in (2) may be replaced by $2\pi i$ times the sum of the residues at the poles of the integrand. Thus finally

$$I = \frac{1}{2nL}\frac{\sin n(t+\frac{1}{2}T)}{\cos\frac{1}{2}nT}+\frac{2}{LT}\sum_{r=0}^{\infty}\frac{\cos(2r+1)\pi t/T}{[n^2-(2r+1)^2\pi^2/T^2]}.$$

## EXAMPLES ON CHAPTER II

1. E.M.F. $E\cos(\omega t+\alpha)$ is applied at $t = 0$ to a circuit consisting of capacity $C$ and inductance $L$ in series. The initial current and charge are zero. Show that the current at time $t$ is

$$E\{\omega\sin(\omega t+\alpha)-n\cos\alpha\sin nt-\omega\sin\alpha\cos nt\}/L(\omega^2-n^2),$$

where $n^2 = 1/LC$, supposed not equal to $\omega^2$.

2. E.M.F. of the resonance frequency, $E\sin nt$, is applied at $t = 0$ to a circuit consisting of capacity $C$ and inductance $L$ in series. The initial current and charge are zero. Show that the current in the circuit at time $t$ is $(E/2L)t\sin nt$, where $n^2 = 1/LC$.

3. E.M.F. $E\sin(\omega t+\alpha)$ is applied at $t = 0$ to an inductive resistance $L$, $R$. The initial current is zero. Show that the current at time $t$ is given by

$$E\{\sin(\gamma-\alpha)e^{-Rt/L}+\sin(\omega t+\alpha-\gamma)\}(R^2+L^2\omega^2)^{-\frac{1}{2}},$$

where $\tan\gamma = L\omega/R$.

† If $n = (2r+1)\pi/T$ for some integer $r$, the circuit is in resonance with one of the harmonics of the applied E.M.F. There are then double poles at $\lambda = \pm in$ and a separate calculation must be made.

4. E.M.F. $E_1$ for $0 < t < T$, $E_2$ for $t > T$, $E_1$ and $E_2$ being constants, is applied to a circuit consisting of $L$, $R$, $C$ in series, the initial current and charge being zero. Show that the current for $t > T$ is given by

$$(E_1/nL)e^{-\mu t}\sin nt - [(E_1 - E_2)/nL]e^{-\mu(t-T)}\sin n(t-T),$$

where $n$ and $\mu$ are defined in § 14, Ex. 1, and it is supposed that $n^2 > 0$.

5. E.M.F. $V = \sin\omega t$, $0 < t < \pi/\omega$, $V = 0$, $t > \pi/\omega$ is applied at $t = 0$ to a circuit consisting of $L$, $R$, $C$ in series, the initial charge and current being zero. Show that the current for $t > \pi/\omega$ is given by

$$-(E/nZL^{\frac{1}{2}}C^{\frac{1}{2}})e^{-\mu t}\{e^{\mu\pi/\omega}\sin[nt - \delta - n\pi/\omega] + \sin(nt - \delta)\},$$

the notation being that of § 14, Ex. 2.

6. Alternating E.M.F. $Ee^{-\mu t}\sin(nt + \alpha)$, of the same period and damping as the circuit, is applied at $t = 0$ to a circuit consisting of $L$, $R$, $C$ in series. The initial charge and current are zero. Show that the current at any subsequent time is

$$Ee^{-\mu t}\{\sin(\alpha - \gamma)\sin nt + nt\sin(nt + \alpha + \gamma)\}/(2n^2L^{\frac{3}{2}}C^{\frac{1}{2}}),$$

where $\tan\gamma = \mu/n$, $n$ and $\mu$ are defined in § 14, Ex. 1, and $n^2$ is supposed positive.

7. Show that a combination of capacity $C$ shunted by resistance $R$, in series with a combination of inductance $L$ shunted by resistance $R$, behaves as a pure resistance for all forms of applied E.M.F. if $L = CR^2$.

8. A condenser of capacity $C_1$, charged to potential $E$, is discharged at $t = 0$ through a resistance $R$ in series with a leaky condenser of capacity $C$ and leakage conductance $G$, initially uncharged. Show that the potential across the condenser $C$ at time $t$ is

$$-(E/KRC)e^{-kt}\sinh Kt,$$

where $k = \frac{1}{2}\left(\frac{1}{RC_1} + \frac{G}{C} + \frac{1}{RC}\right)$, $K^2 = k^2 - G/RCC_1$.

9. A circuit consists of an inductive resistance $L_1$, $R_1$ in series with a combination of $R$, $L$, $C$ in series. The latter combination can be short-circuited by a switch $S$. At $t = 0$, when steady current $E/R_1$ from a battery of E.M.F. $E$ is flowing through the circuit with $S$ closed, the switch $S$ is opened. Show that the subsequent current is

$$[E/nR_1(L + L_1)]e^{-\mu t}[nL_1\cos nt + (R_1 - L_1\mu)\sin nt],$$

where $\mu = \frac{1}{2}(R + R_1)/(L + L_1)$, $n^2 = -\mu^2 + 1/C(L + L_1)$, and $n^2$ is supposed positive.

10. A circuit consists of resistance $R$ in series with a parallel combination of inductive resistance $L_1$, $R_1$ and an inductive resistance $L_2$, $R_2$. There is a switch $S$ in the arm $L_2R_2$ which is open at $t = 0$. At

$t = 0$ a battery of E.M.F. $E$ is applied to the circuit and at $t = T$ the switch $S$ is closed. Show that the battery current for $t > T$ is

$$\frac{E(R_1+R_2)}{R_1 R_2+R(R_1+R_2)} - \frac{E}{L_1 L_2(\alpha_1-\alpha_2)} \sum_{r=1}^{2} (-1)^r e^{\alpha_r(t-T)} \times$$

$$\times \left\{[R_1+R_2+\alpha_r(L_1+L_2)]\Big/\alpha_r + \frac{L_1(R_2+L_2\alpha_r)}{R+R_1}(1-e^{-(R_1+R_2)T/L_1})\right\},$$

where $\alpha_1$, $\alpha_2$ are the roots of

$$L_1 L_2 p^2+[L_1 R_2+R_1 L_2+R(L_1+L_2)]p+R_1 R_2+R(R_1+R_2) = 0.$$

11. Steady-state current due to E.M.F. $E\sin\omega t$ is flowing in a circuit consisting of $L$, $R$, $C$ in series, when at $t = 0$ the resistance $R$ is short-circuited. Show that the current for $t > 0$ is given by

$$\frac{\omega E}{L(\omega^2-n_1^2)}(\cos n_1 t-\cos\omega t)-\frac{E}{Z}\sin\gamma\cos n_1 t+\frac{En_1}{\omega Z}\cos\gamma\sin n_1 t,$$

where $n_1^2 = 1/LC$, and $Z$ and $\gamma$ are defined in § 14, Ex. 2.

12. Alternating current due to E.M.F. $V = E_0\sin(\omega t+\phi)$ is flowing in a circuit consisting of an inductive resistance $L$, $R$, shunted by a leaky condenser of capacity $C$ and leakage conductance $G$. At $t = 0$, when steady-state conditions are supposed to have been attained, the E.M.F. is disconnected. Show that the subsequent potential difference across the condenser is

$$E_0 e^{-\alpha t}\sin\phi\cos\beta t+(E_0/\beta LCZ)[\tfrac{1}{2}Z(RC-LG)\sin\phi-L\sin(\phi-\gamma)]e^{-\alpha t}\sin\beta t,$$

where $\alpha = \frac{1}{2}\left(\frac{R}{L}+\frac{G}{C}\right)$, $\beta^2 = \frac{RG+1}{LC}-\alpha^2$, $Z^2 = R^2+L^2\omega^2$, $\tan\gamma = L\omega/R$.

13. A circuit consists of a parallel combination of capacity $C$ and an inductive resistance $L$, $R$. The initial currents and charges are zero. Constant *current* $I_0$ is fed in across the terminals for $t > 0$. Show that the potential drop across the condenser is given by

$$I_0 R-I_0 Re^{-\mu t}\left[\cos nt-\frac{\mathrm{I}}{n}\left(\frac{1}{RC}-\frac{R}{2L}\right)\sin nt\right],$$

where $n$ and $\mu$ are defined in § 14, Ex. 1.

14. A circuit consists of a combination of capacity $C$, resistance $1/G$, and inductive resistance $L$, $R$, in parallel. At $t = 0$, when the currents and charge are zero, *current* $I_0\sin(\omega t-\alpha)$ is supplied to the circuit. Show that the transform of the voltage across the circuit is given by

$$\overline{V} = \frac{I_0(Lp+R)(\omega\cos\alpha-p\sin\alpha)}{(p^2+\omega^2)[LCp^2+(LG+RC)p+(RG+1)]},$$

and evaluate $V$.

15. Two resistanceless circuits $L_1$, $C_1$ and $L_2$, $C_2$ are coupled by mutual inductance $M$. If at $t = 0$, when the currents and charges are zero,

a battery of E.M.F. $E_0$ is applied in the primary, show that the secondary current is given by

$$\frac{ME_0}{(L_1 L_2-M^2)(\beta^2-\alpha^2)}(\alpha\sin\alpha t-\beta\sin\beta t),$$

where $-\alpha^2$ and $-\beta^2$ are the roots of

$$(L_1 L_2-M^2)x^2+\left(\frac{L_1}{C_2}+\frac{L_2}{C_1}\right)x+\frac{1}{C_1 C_2}=0.$$

16. If E.M.F. $\sin\omega t$ is applied at $t=0$ in the primary circuit of Ex. 15, show that the secondary current is

$$\frac{\omega M}{L_1 L_2-M^2}\left\{\frac{\alpha^2}{(\beta^2-\alpha^2)(\omega^2-\alpha^2)}\cos\alpha t+\right.$$
$$\left.+\frac{\beta^2}{(\beta^2-\alpha^2)(\beta^2-\omega^2)}\cos\beta t+\frac{}{(\omega^2-\alpha^2)(\omega^2-\beta^2)}\cos\omega t\right\}$$

in the notation of Ex. 15. It is assumed $\omega^2\neq\alpha^2$ or $\beta^2$.

17. A circuit $L_1$, $R_1$ is coupled by mutual inductance $M$ to a secondary circuit consisting of $R$, $L$, $C$ in series. At $t=0$, when steady current $E_0/R_1$ is flowing in it, the primary circuit is opened. Show that the subsequent secondary current is given by

$$ME_0\,e^{-\mu t}(n\cos nt-\mu\sin nt)/nR_1 L,$$

where $\mu=R/2L$, $n^2=1/(LC)-\mu^2$, supposed $>0$.

18. Two equal circuits, each consisting of $R$, $L$, $C$ in series, are coupled by mutual inductance $M$. At $t=0$, when the currents and charges are zero, constant E.M.F. $E_0$ is applied in one circuit. Show that the current in the other is given by

$$\tfrac{1}{2}E_0\left\{\frac{1}{(L+M)n_1}e^{-\mu_1 t}\sin n_1 t-\frac{1}{(L-M)n_2}e^{-\mu_2 t}\sin n_2 t\right\},$$

where $\mu_1=\tfrac{1}{2}R/(L+M)$, $\mu_2=\tfrac{1}{2}R/(L-M)$, $n_1^2=-\mu_1^2+1/C(L+M)$, $n_2^2=-\mu_2^2+1/C(L-M)$, provided $n_1^2$ and $n_2^2$ are positive.

19. Two equal circuits, each consisting of $R$, $L$, $C$ in series, are coupled by mutual inductance $M$. There is 'perfect' coupling so that $M=L$. At $t=0$, when the initial currents and charges are all zero, a constant E.M.F. $E_0$ is applied in one circuit. Show that the current in the other is given by

$$(E_0/4Ln)e^{-Rt/4L}\sin nt-(E_0/2R)e^{-t/RC},$$

where $n^2=(1/2LC)-(R^2/16L^2)$, supposed positive.

20. Points $A_1$ and $A_2$, $A_2$ and $A_3$, $A_3$ and $A_4$ are joined by equal resistances $R$. $A_1$, $A_2$, $A_3$, $A_4$ are each joined to $A_5$ by equal capacities $C$. At $t=0$ the condensers in $A_2A_5$, $A_3A_5$, and $A_4A_5$ have no charge and the condenser in $A_1A_5$, which has charge $Q_0$, is discharged into the network. Show that the subsequent charge on the condenser $A_4A_5$ is

$$\tfrac{1}{4}Q_0+\tfrac{1}{4}Q_0\,e^{-2t/RC}-\frac{}{4(2+\sqrt{2})}e^{-(2+\sqrt{2})t/RC}-\frac{Q_0}{4(2-\sqrt{2})}e^{-(2-\sqrt{2})t/RC}.$$

21. The arms $A_1A_2$, $A_2A_3$, $A_3A_4$, $A_4A_1$ of a Wheatstone bridge contain an inductive resistance $L$, $R_1$, and resistances $R_2$, $R_4$, $R_3$ respectively. The galvanometer, of resistance $R_g$, joins $A_2A_4$. The bridge is balanced for steady currents and steady current is flowing from a battery of E.M.F. $E$ connected to $A_1A_3$. Show that if the battery circuit is broken a quantity of electricity

$$\frac{EL(R_2+R_4)}{(R_1+R_2)[(R_1+R_2+R_3+R_4)R_g+(R_1+R_3)(R_2+R_4)]}$$

will flow through the galvanometer.

22. The arms $A_1A_2$, $A_2A_3$, $A_3A_4$, $A_4A_1$ of a Wheatstone bridge contain, respectively, inductive resistance $L_1$, $R_1$, resistance $R_2$, resistance $R_4$, and inductive resistance $L_2$, $R_3$. $A_2A_4$ contains the galvanometer. There is mutual inductance $M$ between the arms $A_1A_3$ and $A_1A_2$. E.M.F. $V$ is applied in $A_1A_3$. Show that there will be no galvanometer current at any time provided

$$\frac{L_1+M}{L_2-M} = \frac{R_2}{R_4} = \frac{R_1}{R_3}.$$

23. The arms $A_1A_2$, $A_2A_3$, $A_3A_4$, $A_4A_1$ of a Wheatstone bridge contain a combination of inductive resistance $L$, $R_1$ shunted by a condenser $C$, resistance $R_2$, resistance $R_4$, and resistance $R_3$ respectively. $A_2A_4$ contains the galvanometer. Show that

(i) there is a balance for steady-state alternating E.M.F. connected to $A_1A_3$ if $LR_4 = CR_1R_2R_3$ and $R_2R_3LC\omega^2 = R_2R_3-R_1R_4$;

(ii) if the bridge is balanced for direct current, and steady current is flowing from a battery of E.M.F. $E$ connected to $A_1A_3$, there is a ballistic balance when the battery circuit is opened if $L = CR_1^2$.

24. For the filter circuit of Fig. 9, with $z = R$, $z' = Lp+R'$, $z'' = 0$, $\bar{V} = E/p$, show that the current in the $r$th section is given by

$$\frac{E}{mR'}[1-e^{-R't/L}]+\frac{(-1)^rE}{m(4R+R')}[1-e^{-(4R+R')t/L}]+$$
$$+\frac{2E}{Lm}\sum_{s=1}^{m-1}\frac{1}{\alpha_s}\cos\frac{rs\pi}{m}[1-e^{-\alpha_s t}],$$

where $\alpha_s = [2R(1-\cos s\pi/m)+R']/L$.

25. For the filter circuit of Fig. 9, with $z' = R'+1/C'p$, $z = 1/Cp$, $z'' = 0$, $\bar{V} = E/p$, show that the current in the $r$th section is given by

$$\frac{E}{mR'}e^{-t/R'C'}+\frac{(-1)^rE}{mR'}e^{-t(C+4C')/R'CC'}+\frac{2E}{mR'}\sum_{s=1}^{m-1}\cos\frac{rs\pi}{m}e^{-\alpha_s t},$$

where $\alpha_s = 2[(C/2C')+1-\cos s\pi/m]/R'C$.

26. For the filter circuit of Fig. 9, with $z' = R'+1/C'p$, $z = R$, $z'' = 0$, $\bar{V} = E/p$, show that the current in the $r$th section is given by

$$\frac{E}{mR'}e^{-t/R'C'}+\frac{(-1)^r E}{m(R'+4R)}e^{-t/C'(R'+4R)}+\frac{2EC'}{m}\sum_{s=1}^{m-1}\alpha_s e^{-\alpha_s t},$$

where $\alpha_s = 1/[R'C'+2RC'(1-\cos s\pi/m)]$.

27. For the filter circuit of Fig. 9, with $z' = R$, $z = 1/Cp$, $z'' = 0$, $V = \sin\omega t$, i.e. E.M.F. $\sin\omega t$ applied to the circuit of § 19, Ex. 1, show that the *transient current* in the $r$th section is

$$-\frac{4\omega C(-1)^r}{m(\omega^2R^2C^2+16)}e^{-4t/RC}-\frac{2\omega}{mR}\sum_{s=1}^{m-1}\frac{\alpha_s}{\omega^2+\alpha_s^2}\cos\frac{rs\pi}{m}e^{-\alpha_s t},$$

where $\alpha_s = (2/RC)(1-\cos s\pi/m)$.

## CHAPTER III

# DYNAMICAL APPLICATIONS

In this chapter a few examples will be given to illustrate the application of the method to dynamical problems. Roughly it may be said that whenever such a problem leads to an ordinary linear differential equation with constant coefficients which has to be solved with given initial conditions, the Laplace Transformation provides a simple method of solution.

**22.** *Two flywheels of moments of inertia $I_1$ and $I_2$ are connected by an elastic shaft of negligible moment of inertia. The whole system is rotating with constant angular velocity $\omega$ when at $t = 0$ a constant retarding couple $P$ is applied to the wheel $I_1$. It is required to find the subsequent angular velocity of the wheel $I_2$.*

Let $\theta_1$ and $\theta_2$ be the angular displacements of the wheels, then we may take $\theta_1 = \theta_2 = 0$, $D\theta_1 = D\theta_2 = \omega$, when $t = 0$. Let $\lambda$ be the stiffness of the shaft, i.e. the couple per radian relative twist of the wheels. Then the equations of motion are

$$\left.\begin{aligned} I_1 D^2\theta_1-\lambda(\theta_2-\theta_1) &= -P, \\ I_2 D^2\theta_2+\lambda(\theta_2-\theta_1) &= 0, \end{aligned}\right\} \quad t > 0, \tag{1}$$

to be solved with the above initial conditions.

The subsidiary equations are

$$\begin{aligned} (I_1 p^2+\lambda)\bar\theta_1-\lambda\bar\theta_2 &= I_1\omega-P/p, \\ -\lambda\bar\theta_1+(I_2 p^2+\lambda)\bar\theta_2 &= I_2\omega. \end{aligned} \tag{2}$$

Hence $$\bar\theta_2 = \frac{\omega}{p^2}-\frac{\lambda P}{p^3[I_1 I_2 p^2+\lambda(I_1+I_2)]}.$$

If $\phi$ is the angular velocity of $I_2$, we have

$$\bar\phi = p\bar\theta_2 = \frac{\omega}{p}-\frac{\lambda P}{I_1 I_2 p^2(p^2+n^2)}$$

$$= \frac{\omega}{p}-\frac{\lambda P}{I_1 I_2 n^2}\left\{\frac{1}{p^2}-\frac{1}{p^2+n^2}\right\}, \tag{3}$$

where $n^2 = \lambda(I_1+I_2)/I_1 I_2$.

Hence $$\phi = \omega-\frac{Pt}{I_1+I_2}+\frac{P}{n(I_1+I_2)}\sin nt. \tag{4}$$

*As another example, suppose the retarding couple $P$ is applied for time $T$ only.*

Here the couple which appears in the right-hand side of the first equation of (1) is

$$\left.\begin{array}{ll} -P, & 0 < t < T, \\ 0, & t > T. \end{array}\right\}$$

Thus its Laplace Transform is

$$-P\int\limits_0^T e^{-pt}\,dt = -\frac{P}{p}(1-e^{-pT}) \tag{5}$$

and the only change is that $-P/p$ in (2) and (3) is to be replaced by (5). Thus

$$\bar\phi = \frac{\omega}{p}-\frac{\lambda P}{I_1 I_2 n^2}\left(\frac{1}{p^2}-\frac{1}{p^2+n^2}\right)(1-e^{-pT}).$$

Then from Theorem IV we see that $\phi$ is given by (4) when $0 < t < T$, while for $t > T$ it is given by

$$\omega-\frac{Pt}{I_1+I_2}+\frac{P\sin nt}{n(I_1+I_2)}+\frac{P(t-T)}{I_1+I_2}-\frac{P\sin n(t-T)}{n(I_1+I_2)}$$

$$= \omega-\frac{PT}{I_1+I_2}+\frac{2P}{n(I_1+I_2)}\cos n(t-\tfrac{1}{2}T)\sin \tfrac{1}{2}nT.$$

**23.** *A particle of mass $m$ is hung by an elastic string of length $l$ and modulus of elasticity $\lambda$ from a point vertically above, and is at rest with the string unstretched. At $t = 0$ the point of support commences to make a vertical oscillation $a\sin\omega t$ about its original position and the particle is released. To find the subsequent motion of the particle.*

Let $\xi$ and $x$, both measured downwards, be the displacements of the point of support and the particle from the original position of the point of support. Then the equation of motion is

$$mD^2x = mg-\lambda(x-\xi-l)/l,$$

or
$$D^2x+n^2x = n^2a\sin\omega t+g+n^2l, \tag{1}$$

where $n^2 = \lambda/ml$.

This is to be solved with $x = l$, $Dx = 0$, when $t = 0$. The subsidiary equation is

$$(p^2+n^2)\bar{x} = \frac{n^2 a\omega}{p^2+\omega^2}+pl+\frac{g+n^2 l}{p}.$$

Thus, if $n \neq \omega$,

$$\bar{x} = \frac{n^2 a\omega}{(p^2+n^2)(p^2+\omega^2)}+\frac{pl}{p^2+n^2}+\frac{g+n^2 l}{p(p^2+n^2)}$$

$$= \frac{n^2 a\omega}{\omega^2-n^2}\left[\frac{1}{p^2+n^2}-\frac{1}{p^2+\omega^2}\right]+\frac{pl}{p^2+n^2}+\frac{g+n^2 l}{n^2}\left[\frac{1}{p}-\frac{p}{p^2+n^2}\right].$$

Therefore

$$x = \frac{na}{\omega^2-n^2}(\omega \sin nt - n \sin \omega t)+\frac{g+n^2 l}{n^2}-\frac{g}{n^2}\cos nt.$$

If $\omega = n$,

$$x = \tfrac{1}{2}a(\sin nt - nt\cos nt)+\frac{g+n^2 l}{n^2}-\frac{g}{n^2}\cos nt.$$

**24.** *To find the motion of a particle of charge e and mass m acted on by an electric field E parallel to OX, and by a magnetic field H parallel to OZ. The particle is projected at $t = 0$ from the origin with velocity $(u, v, w)$.*

The equations of motion are

$$mD^2x = Ee+\frac{eH}{c}Dy,$$

$$mD^2y = -\frac{eH}{c}Dx,$$

$$mD^2z = 0,$$

and so, with the above initial conditions, we have the subsidiary equations

$$mp^2\bar{x}-\frac{eH}{c}p\bar{y} = \frac{Ee}{p}+mu,$$

$$mp^2\bar{y}+\frac{eH}{c}p\bar{x} = mv,$$

$$mp^2\bar{z} = mw,$$

Solving, we have

$$\bar{x} = \frac{\alpha(Hv+cE)}{Hp(p^2+\alpha^2)}+\frac{u}{p^2+\alpha^2},$$

$$\bar{y} = \frac{v}{p^2}-\frac{\alpha^2(Hv+cE)}{Hp^2(p^2+\alpha^2)}-\frac{\alpha u}{p(p^2+\alpha^2)},$$

$$z = \frac{w}{p^2},$$

where $\alpha = eH/mc$.

And so, using

$$\frac{1}{p(p^2+\alpha^2)} = \frac{1}{\alpha^2}\left\{\frac{1}{p}-\frac{p}{p^2+\alpha^2}\right\}$$

and

$$\frac{1}{p^2(p^2+\alpha^2)} = \frac{1}{\alpha^2}\left\{\frac{1}{p^2}-\frac{1}{p^2+\alpha^2}\right\},$$

we have

$$x = \frac{(Hv+cE)}{H\alpha}(1-\cos\alpha t)+\frac{u}{\alpha}\sin\alpha t,$$

$$y = vt-\frac{(Hv+cE)}{H\alpha}(\alpha t-\sin\alpha t)-\frac{u}{\alpha}(1-\cos\alpha t),$$

$$z = wt.$$

**25.** *Motion of a projectile relative to the earth.*

We find the path of a particle, projected with velocity $(u, v, w)$ from the origin in latitude $\lambda$: if the $Z$-axis is in the direction of apparent gravity, the $X$-axis to the East, and the $Y$-axis to the North, the equations of motion are†

$$\left.\begin{aligned} D^2x-2\omega Dy\sin\lambda+2\omega Dz\cos\lambda &= 0, \\ D^2y+2\omega Dx\sin\lambda &= 0, \\ D^2z-2\omega Dx\cos\lambda &= -g, \end{aligned}\right\}$$

where $\omega$ is the earth's angular velocity.

Thus, with the initial conditions above, the subsidiary equations are

$$\left.\begin{aligned} p^2\bar{x}-2\omega p\bar{y}\sin\lambda+2\omega p\bar{z}\cos\lambda &= u, \\ p^2\bar{y}+2\omega p\bar{x}\sin\lambda &= v, \\ p^2\bar{z}-2\omega p\bar{x}\cos\lambda &= w-g/p. \end{aligned}\right\}$$

† Lamb, *Higher Mechanics*, 2nd ed. (Cambridge, 1929), § 66.

Solving, we have

$$\bar{x} = \frac{2\omega(v\sin\lambda - w\cos\lambda)}{p(p^2+4\omega^2)} + \frac{2g\omega\cos\lambda}{p^2(p^2+4\omega^2)} + \frac{u}{p^2+4\omega^2},$$

$$\bar{y} = -\frac{4\omega^2\sin\lambda(v\sin\lambda - w\cos\lambda)}{p(p^2+4\omega^2)} - \frac{4g\omega^2\sin\lambda\cos\lambda}{p^3(p^2+4\omega^2)} - \frac{2\omega u\sin\lambda}{p(p^2+4\omega^2)} + \frac{v}{p^2},$$

$$\bar{z} = \frac{4\omega^2\cos\lambda(v\sin\lambda - w\cos\lambda)}{p^2(p^2+4\omega^2)} + \frac{4g\omega^2\cos^2\lambda}{p^3(p^2+4\omega^2)} + \frac{2\omega u\cos\lambda}{p(p^2+4\omega^2)} + \frac{w}{p^2} - \frac{g}{p^3}.$$

Therefore

$$x = \frac{v\sin\lambda - w\cos\lambda}{2\omega}(1-\cos 2\omega t) + \frac{g\cos\lambda}{4\omega^2}(2\omega t - \sin 2\omega t) + \frac{u}{2\omega}\sin 2\omega t,$$

$$y = -\frac{\sin\lambda(v\sin\lambda - w\cos\lambda)}{2\omega}(2\omega t - \sin 2\omega t) - \frac{g\sin\lambda\cos\lambda}{4\omega^2}(2\omega^2t^2 - 1 + \cos 2\omega t) - \frac{u\sin\lambda}{2\omega}(1-\cos 2\omega t) + vt,$$

$$= \frac{\cos\lambda(v\sin\lambda - w\cos\lambda)}{2\omega}(2\omega t - \sin 2\omega t) + \frac{g\cos^2\lambda}{4\omega^2}(2\omega^2t^2 - 1 + \cos 2\omega t) + \frac{u\cos\lambda}{2\omega}(1-\cos 2\omega t) + wt - \tfrac{1}{2}gt^2.$$

**26.** *Small oscillations about equilibrium using the Lagrange equations.*†

Consider a system of $n$ degrees of freedom in which the kinetic and potential energies are expressible in the form

$$2T = \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}\dot{q}_r\dot{q}_s,$$

$$2V = \sum_{r=1}^{n}\sum_{s=1}^{n} c_{rs}q_r q_s.$$

† In this section and the next dots will be used for differentiation with respect to the time.

The Lagrange equations† are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r} = -\frac{\partial V}{\partial q_r}, \quad r = 1,\ldots, n.$$

For small oscillations about equilibrium—in which case terms of the second order in coordinates and velocities may be neglected and the $a_{rs}$ taken as constants with the values they have in the equilibrium position—these become

$$\sum_{s=1}^{n} (a_{rs}\ddot{q}_s+c_{rs}q_s) = 0, \quad r = 1,\ldots, n. \tag{1}$$

This is a system of $n$ ordinary linear differential equations with constant coefficients which has to be solved with the values of $q_r$ and $\dot{q}_r$ $(r = 1,\ldots, n)$ given when $t = 0$; it may be solved by the methods of Chapter I.

The following properties of the coefficients $a_{rs}$ and $c_{rs}$ are known from general dynamical theory:‡

$$\text{for all } r \text{ and } s, \tag{2}$$

$$A \equiv \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \neq 0. \tag{3}$$

It follows from (3) that exceptional cases of the type discussed in § 9 cannot occur.

If there are in addition resistances to the motion proportional to the velocities, equation (1) is replaced by||

$$\sum_{s=1}^{n} (a_{rs}\ddot{q}_s+b_{rs}\dot{q}_s+c_{rs}q_s) = 0, \quad r = 1,\ldots, n, \tag{4}$$

where (2) and (3) still hold and

$$b_{rs} = b_{sr}. \tag{5}$$

For a cyclic system†† the form (4) again holds with (2) and (3), but in place of (5)

$$b_{rs} = -b_{sr}. \tag{6}$$

In problems on forced oscillations terms depending only on the time will appear in the right-hand sides of equations (4).

All these cases can be treated immediately by the methods

† Lamb, loc. cit., § 77. ‡ Ibid., § 73. || Ibid., § 98. †† Ibid., § 99.

of Chapter I. No further theory is necessary. As an example consider the following:

*Particles of mass $3m$, $4m$, $3m$ are equally spaced along a string of length $4l$, fixed at its ends and stretched to tension $T$. At $t = 0$, when the system is at rest in the equilibrium position, a transverse impulse $I$ is given to the first particle. Find the subsequent motion.*

Let $x_1$, $x_2$, $x_3$ be the displacements of the particles from their equilibrium positions. Then the kinetic energy of the system is

$$\tfrac{1}{2}m(3\dot{x}_1^2+4\dot{x}_2^2+3\dot{x}_3^2),$$

and its potential energy is†

$$(T/l)(x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3).$$

The Lagrange equations are

$$\left.\begin{aligned} 3\ddot{x}_1+n^2(2x_1-x_2) &= 0,\\ 4\ddot{x}_2+n^2(2x_2-x_1-x_3) &= 0,\\ 3\ddot{x}_3+n^2(2x_3-x_2) &= 0, \end{aligned}\right\} \tag{7}$$

where $n^2 = T/ml$. These are to be solved with

$$x_1 = x_2 = x_3 = \dot{x}_2 = \dot{x}_3 = 0, \quad \dot{x}_1 = I/3m, \quad \text{when } t = 0. \tag{8}$$

The subsidiary equations are

$$\left.\begin{aligned} (3p^2+2n^2)\bar{x}_1-n^2\bar{x}_2 &= I/m,\\ -n^2\bar{x}_1+(4p^2+2n^2)\bar{x}_2-n^2\bar{x}_3 &= 0,\\ -n^2\bar{x}_2+(3p^2+2n^2)\bar{x}_3 &= 0. \end{aligned}\right\} \tag{9}$$

Solving successively, we obtain

$$\left.\begin{aligned} \bar{x}_2 &= \frac{n^2 I}{2m(p^2+n^2)(6p^2+n^2)} = \frac{I}{10m}\left\{\frac{6}{6p^2+n^2}-\frac{1}{p^2+n^2}\right\},\\ \bar{x}_3 &= \frac{n^2\bar{x}_2}{3p^2+2n^2} = \frac{I}{10m}\left\{\frac{4}{6p^2+n^2}+\frac{1}{p^2+n^2}-\frac{5}{3p^2+2n^2}\right\},\\ \bar{x}_1 &= \frac{I}{m(3p^2+2n^2)}+\frac{n^2\bar{x}_2}{3p^2+2n^2}\\ &= \frac{I}{10m}\left\{\frac{4}{6p^2+n^2}+\frac{1}{p^2+n^2}+\frac{5}{3p^2+2n^2}\right\}. \end{aligned}\right\} \tag{10}$$

† Ibid., § 87.

And therefore

$$
\begin{aligned}
x_1 &= \frac{I}{10mn}\left\{\frac{4}{\sqrt{6}}\sin\frac{nt}{\sqrt{6}}+\sin nt+\frac{5}{\sqrt{6}}\sin nt\right. \\
x_2 &= \frac{I}{10mn}\left\{\sqrt{6}\sin\frac{nt}{\sqrt{6}}-\sin nt\right\}, \\
x_3 &= \frac{I}{10mn}\left\{\frac{4}{\sqrt{6}}\sin\frac{nt}{\sqrt{6}}+\sin nt-\frac{5}{\sqrt{6}}\sin nt\sqrt{\left(\frac{2}{3}\right)}\right\}.
\end{aligned} \tag{11}
$$

**27.** *Comparison with the method of normal coordinates.*

The classical method of solving equations § 26 (4), namely,

$$\sum_{s=1}^{n}(a_{rs}\ddot{q}_s+b_{rs}\dot{q}_s+c_{rs}q_s) = 0, \quad r = 1,\dots, n, \tag{1}$$

is to seek a solution of type $q_s = \lambda_s e^{\alpha t}$, $s = 1,\dots, n$.

Substituting in (1) we obtain the $n$ homogeneous equations

$$\sum_{s=1}^{n}(\alpha^2 a_{rs}+\alpha b_{rs}+c_{rs})\lambda_s = 0, \quad r = 1,\dots, n. \tag{2}$$

The consistency condition for these is

$$D(\alpha) = \begin{vmatrix} \alpha^2 a_{11}+\alpha b_{11}+c_{11} & \cdots & \alpha^2 a_{1n}+\alpha b_{1n}+c_{1n} \\ \vdots & & \vdots \\ \alpha^2 a_{n1}+\alpha b_{n1}+c_{n1} & \cdots & \alpha^2 a_{nn}+\alpha b_{nn}+c_{nn} \end{vmatrix} = 0, \tag{3}$$

which has $2n$ roots $\alpha_1,\dots, \alpha_{2n}$, assumed for the present to be all different.

For each of these roots $\alpha_k$ we can solve the system (2) for the ratios of the $\lambda$, obtaining

$$\lambda_1^{(k)} : \lambda_2^{(k)} : \dots : \lambda_n^{(k)}. \tag{4}$$

Each of these solutions

$$q_r = \lambda_r^{(k)} e^{\alpha_k t}, \quad r = 1,\dots, n, \tag{5}$$

is called a normal mode of motion, and the most general solution is given by a linear combination of them, namely,

$$q_r = \sum_{k=1}^{2n} A_k \lambda_r^{(k)} e^{\alpha_k t}, \quad r = 1,\dots, n, \tag{6}$$

where the $2n$ constants $A_k$ are to be found from the conditions at $t = 0$. If $u_r$ and $v_r$ are the values of $q_r$ and $\dot{q}_r$ for $t = 0$, we

have, when $t = 0$,

$$\left.\begin{aligned} \sum_{k=1}^{2n} A_k \lambda_r^{(k)} &= u_r, \quad r = 1,\ldots, n, \\ \sum_{k=1}^{2n} \alpha_k A_k \lambda_r^{(k)} &= v_r, \quad r = 1,\ldots, n. \end{aligned}\right\} \tag{7}$$

Solving these $2n$ equations we obtain the constants $A_1,\ldots, A_{2n}$. The motion† may be regarded as a superposition of vibrations in the normal form with amplitudes $A_1,\ldots, A_{2n}$.

It will be noticed that some algebraical steps remain to be filled in in the above sketch; for example, it has to be shown that the equations (7) are linearly independent. The complete theory is, in fact, not difficult unless the equation $D(\alpha) = 0$ has repeated roots, in which case considerable algebraical difficulties arise.‡ These are absent in the Laplace Transformation solution.

Consider now the solution of the example of § 26 by the method of this section. The equation $D(\alpha) = 0$ is

$$\begin{vmatrix} 3\alpha^2+2n^2 & -n^2 & 0 \\ -n^2 & 4\alpha^2+2n^2 & -n^2 \\ 0 & -n^2 & 3\alpha^2+2n^2 \end{vmatrix} = 0,$$

of which the roots are $\pm in/\sqrt{6}$, $\pm in$, $\pm in\sqrt{\tfrac{2}{3}}$. The ratios of the amplitudes in the three normal modes are respectively

$$\left.\begin{aligned} \lambda_1^{(1)} : \lambda_2^{(1)} : \lambda_3^{(1)} &= 2 : 3 : 2, \\ \lambda_1^{(2)} : \lambda_2^{(2)} : \lambda_3^{(2)} &= 1 : -1 : 1, \\ \lambda_1^{(3)} : \lambda_2^{(3)} : \lambda_3^{(3)} &= 1 : 0 : -1. \end{aligned}\right\} \tag{8}$$

The general linear combination of these terms is

$$\left.\begin{aligned} x_1 &= 2A_1 \sin[A_2+nt/\sqrt{6}]+A_3 \sin[A_4+nt]+ \\ &\qquad\qquad +A_5 \sin[A_6+nt\sqrt{\tfrac{2}{3}}], \\ x_2 &= 3A_1 \sin[A_2+nt/\sqrt{6}]-A_3 \sin[A_4+nt], \\ x_3 &= 2A_1 \sin[A_2+nt/\sqrt{6}]+A_3 \sin[A_4+nt]- \\ &\qquad\qquad -A_5 \sin[A_6+nt\sqrt{\tfrac{2}{3}}]. \end{aligned}\right\} \tag{9}$$

† Routh and Heaviside first developed this method, and Heaviside (*Electrical Papers*, **1**, 523) gave a formula for determining the amplitudes $A_1,\ldots, A_{2n}$. Bromwich, in his classical paper 'On normal coordinates in dynamical systems' (*Proc. London Math. Soc.* (2), **15** (1914), 413), showed that the amplitudes obtained in this way agreed with those found by a method related to that of Chapter I. Heaviside's development of his operational method was subsequent to his study of this method and indeed grew out of it.

‡ Lamb, loc. cit., § 93.

The six constants $A_1,\ldots, A_6$ are to be determined from the six equations $x_1 = x_2 = x_3 = \dot{x}_2 = \dot{x}_3 = 0$, $\dot{x}_1 = I/3m$, when $t = 0$. Solving and substituting gives the result § 26 (11). It will be seen that the algebra of that section is considerably shorter.

The relation between the results obtained by the Laplace Transformation and by the method of normal coordinates may now be seen. For simplicity we consider only the case in which all the $b_{rs}$ vanish; here there corresponds to each root of the period equation (supposed all different) a factor of the denominator of all the Laplace Transforms: these roots occur, in fact, in conjugate imaginary pairs $\pm i\nu_k$, $k = 1, 2,\ldots, n$; thus to each such pair corresponds a partial fraction with quadratic denominator $(p^2+\nu_k^2)$ in the expressions for the transforms. The coefficients of the partial fraction with denominator $p^2+\nu_k^2$ in $\bar{x}_1, \bar{x}_2,\ldots, \bar{x}_n$ are in the ratio $\lambda_1^{(k)} : \lambda_2^{(k)} : \ldots : \lambda_n^{(k)}$ for the normal mode of frequency $\nu_k/2\pi$. Similar results hold when the $b_{rs}$ are not all zero.

## EXAMPLES ON CHAPTER III

1. A particle of mass $m$ moves in a straight line under restoring force $m\lambda$ times the displacement and resistance $2\mu m$ times the velocity. If it is projected at $t = 0$ with velocity $u_0$ at distance $x_0$ from its equilibrium position, show that if $n^2 = \lambda-\mu^2 > 0$ the subsequent displacement is

$$\frac{1}{n}e^{-\mu t}\{nx_0 \cos nt+(u_0+\mu x_0)\sin nt\},$$

and discuss the cases $n^2 \leqslant 0$.

2. A particle of mass $m$ can perform small oscillations about equilibrium under restoring force $mn^2$ times the displacement. It is started from rest in its equilibrium position by a constant force $P$ which acts for time $T$ and then ceases. Show that the amplitude of the subsequent oscillation is

$$\frac{2P}{mn^2}\sin \tfrac{1}{2}nT.$$

3. A mass $m$ rests on a horizontal plane of coefficient of friction $\mu$, and is connected to a fixed point of the plane by a spring of stiffness $\lambda$. It starts from rest at a distance $a$ from the position in which the spring is neither extended nor compressed. Find the motion, and show that the mass will next come to rest at distance $a-2mg\mu/\lambda$ from the equilibrium position.

4. A simple pendulum of length $l$ is set in motion from rest in its equilibrium position by small horizontal motion of its point of support. Show that

(i) if the point of support is displaced a horizontal distance $a$, the displacement of the bob is

$$a(1-\cos nt), \qquad n^2 = g/l;$$

(ii) if the point of support makes half a complete oscillation of $a\sin\omega t$, the displacement is

$$\frac{na}{\omega^2-n^2}(\omega\sin nt - n\sin\omega t), \quad 0 < t < \pi/\omega,$$

and

$$\frac{2na\omega}{\omega^2-n^2}\sin\left(nt-\frac{n\pi}{2\omega}\right)\cos\frac{n\pi}{2\omega}, \quad t > \pi/\omega;$$

(iii) if the point of support makes $r$ complete oscillations of the resonance frequency, the displacement is

$$\tfrac{1}{2}a(\sin nt - nt\cos nt), \quad 0 < t < 2r\pi/n,$$

and

$$-ar\pi\cos nt, \quad t > 2r\pi/n.$$

5. A particle hangs at rest in the equilibrium position at the end of an elastic string whose unstretched length is $a$. The equilibrium length of the string is $b$ and the period of oscillation about equilibrium is $2\pi/n$. At $t = 0$ the point of support begins to move so that its downward displacement at time $t$ is $c\sin\omega t$. Show that the length of the string at time $t$ is

$$b-\frac{cn\omega}{n^2-\omega^2}\sin nt+\frac{c\omega^2}{n^2-\omega^2}\sin\omega t, \quad \text{if } n \neq \omega,$$

or

$$b+\tfrac{1}{2}c\sin nt-\tfrac{1}{2}nct\cos nt, \quad \text{if } n = \omega.$$

6. Two equal masses $m$, free to move in a straight line, are connected by a spring of stiffness $\lambda$. At $t = 0$, when they are both at rest and the spring unstrained, a force $P$ is applied to one of them in the direction towards the other mass. Show that the displacement of the other mass from its initial position is

$$\frac{P}{4m}\left\{t^2-\frac{m}{\lambda}+\frac{m}{\lambda}\cos t\sqrt{(2\lambda/m)}\right\}$$

7. Show that in the problem of Ex. 6, if the force is $P\sin\omega t$, the displacement is given by

$$\frac{P}{2\omega^2mn(n^2-\omega^2)}\{(n^2-\omega^2)\omega nt-n^3\sin\omega t+\omega^3\sin nt\},$$

where $n^2 = 2\lambda/m$, and $\omega^2 \neq n^2$.

8. In the problem of Ex. 6, if the applied force is one half-wave of $\sin\omega t$, i.e. is $P\sin\omega t$ for $0 < t < \pi/\omega$ and zero for $t > \pi/\omega$, show that the displacement is given by the answer of Ex. 7 if $0 < t < \pi/\omega$, and if $t > \pi/\omega$ is

$$\frac{P}{2\omega^2mn(n^2-\omega^2)}\{\omega n(n^2-\omega^2)(2t-\pi/\omega)+2\omega^3\sin n(t-\pi/2\omega)\cos n\pi/2\omega\}$$

if $\omega^2 \neq n^2$. Find also the displacement if $n^2 = \omega^2$.

9. Two particles of masses $M$ and $m$ are connected by a spring of stiffness $\lambda$ and are at rest in equilibrium on a smooth horizontal plane when the particle $M$ is given a blow $P$ in the direction towards the other particle. Show that the subsequent displacement of the particle $M$ is

$$\frac{P}{M(M+m)n}(Mnt+m\sin nt),$$

where $n^2 = \lambda\left(\frac{1}{M}+\frac{1}{m}\right)$.

10. A uniform rod $AB$ of mass $m$ and length $2a$ is supported at its ends by equal springs of stiffness $\lambda$ whose other ends are fixed to a horizontal plane. When the rod is at rest at $t = 0$ in its equilibrium position, a vertical blow $P$ is struck at one end $A$. Show that the subsequent displacement of that end is

$$\frac{P}{m}\left\{\frac{1}{n\sqrt{2}}\sin 2^{\frac{1}{2}}nt+\frac{3}{n\sqrt{6}}\sin 6^{\frac{1}{2}}nt\right\},$$

where $n^2 = \lambda/m$.

11. The rod $AB$ of Ex. 10 is at rest in its equilibrium position at $t = 0$ when the other end of the spring supporting $A$ is given a motion $a\sin\omega t$, $0 < t < \pi/\omega$, zero $t > \pi/\omega$, where $a$ is small. Find the angular displacement of the rod for $t > \pi/\omega$.

12. A particle is projected vertically upwards in latitude $\lambda$ with velocity $V$. Show that owing to the earth's rotation it will strike the ground again at a point

$$\frac{4}{3g^2}\omega V^3\cos\lambda$$

to the West of its starting-point.

13. A particle falls freely under gravity in latitude $\lambda$ from relative rest at the origin. Show that if the earth's rotation is taken into account its displacement towards the East at time $t$ is

$$\tfrac{1}{4}g\cos\lambda(2\omega t-\sin 2\omega t)/\omega^2.$$

Find the Easterly deviation in a fall of 100 metres and show that the Northerly deviation is relatively negligible.

14. In latitude 45° N. a gun is fired due North at an object distant 20 kilometres, this being the maximum range of the gun. Show that, if the earth's rotation has not been allowed for in aiming, the shell should fall about 44 metres East of the mark. Show also that if the shell is fired South under similar conditions the deviation will be twice as great and towards the West. (Air-resistance is neglected.)

15. A straight, imperfectly rough tube starts to rotate at $t = 0$ in a horizontal plane about one end $O$ with angular velocity $\omega$. The tube contains a particle which is initially at distance $a$ from $O$ and at rest relative to the tube. Prove that at time $t$ the distance of the particle from $O$ is

$$ae^{-\mu\omega t}\{\mu(1+\mu^2)^{-\frac{1}{2}}\sinh\omega t(1+\mu^2)^{\frac{1}{2}}+\cosh\omega t(1+\mu^2)^{\frac{1}{2}}\}.$$

16. A particle is projected vertically upwards at $t = 0$ with velocity $V$ from the origin under gravity and resistance $2km$ times the velocity. Show that its displacement at time $t$ is

$$-\frac{gt}{2k}+\frac{(g+2kV)}{4k^2}(1-e^{-2kt}).$$

17. A particle travels in a resisting medium which produces a retardation $2\lambda V$, where $V$ is the velocity, and it is attracted to the origin with an intensity $\mu^2 r$. It is projected from $(a, 0)$ with a velocity $v$ parallel to $Oy$. Prove that, if $\mu > \lambda$, the orbit is

$$x = \frac{ae^{-\lambda t}}{\cos\alpha}\cos(\mu t\cos\alpha-\alpha),$$

$$y = \frac{\quad}{\mu\cos\alpha}\sin(\mu t\cos\alpha),$$

where $\sin\alpha = \lambda/\mu$.

Find the orbit when $\mu < \lambda$.

18. A particle of mass $m$ and charge $e$ is projected from the origin with velocity $(u, 0, 0)$ and is subject to magnetic field $H$ along the $Z$-axis and resistance to motion $km$ times the velocity. Show that its coordinates at time $t$ are

$$\frac{ku}{\lambda^2+k^2}-\frac{ku}{\lambda^2+k^2}e^{-kt}\cos\lambda t+\frac{\lambda u}{\lambda^2+k^2}e^{-kt}\sin\lambda t,$$

$$y = \frac{\lambda u}{\lambda^2+k^2}+\frac{u}{\lambda^2+k^2}e^{-kt}(\lambda\cos\lambda t+k\sin\lambda t),$$

where $\lambda = eH/mc$.

19. A particle of mass $m$ and charge $e$ is projected from the origin at $t = 0$ with velocity $(u, v, w)$ and is subject to an electric field $E\sin(\omega t+\alpha)$ along the $X$-axis and a magnetic field $H$ along the $Z$-axis. Show that the $x$-coordinate of the particle at time $t$ is

$$\frac{v}{\lambda}-\frac{v}{\lambda}\cos\lambda t+\frac{u}{\lambda}\sin\lambda t+\frac{Ee}{m(\omega^2-\lambda^2)}\left\{\sin\alpha\cos\lambda t+\frac{\omega}{\lambda}\cos\alpha\sin\lambda t-\sin(\omega t+\alpha)\right\}.$$

20. A particle of mass $m$ and charge $e$ is acted on by an electric field $E$ parallel to $OX$ and by a magnetic field $H$ parallel to $OZ$ and resistance to motion $m\mu$ times the velocity. It is released from the origin at $t = 0$ with zero velocity. Show that its subsequent displacement parallel to $OX$ is

$$\frac{Ee}{m}\left\{\frac{\alpha^2-\mu^2}{(\mu^2+\alpha^2)^2}+\frac{\mu t}{(\mu^2+\alpha^2)}\quad\frac{\ ^2-\alpha^2)}{(\mu^2+\alpha}\,e^{-\mu t}\cos\alpha t-\frac{2\alpha\mu}{(\mu^2+\alpha^2)^2}e^{-\mu t}\sin\alpha t\right\},$$

where $\alpha = eH/mc$.

21. A machine is of total mass $M$ and is supported symmetrically on four springs each of stiffness $\lambda$. The reciprocating part of the machine is of mass $m$ and moves vertically in simple harmonic motion of amplitude $a$. There is resistance to motion of the machine $2Mk$ times the

velocity. If the machine starts to work at $t = 0$ with frequency $\omega/2\pi$ from rest in the equilibrium position, show that the subsequent displacement of the bed is

$$\frac{ma^2}{Mn_1\{(n^2-\omega^2)^2+4k^2\omega^2\}}\{n_1(n^2-\omega^2)\sin\omega t-2k\omega n_1\cos\omega t+$$
$$+2k\omega n_1 e^{-kt}\cos n_1 t+\omega(\omega^2+2k^2-n^2)e^{-kt}\sin n_1 t\},$$

where $n^2 = 4\lambda/M$, $n_1^2 = n^2-k^2$, supposed positive.

22. A light string of length $3l$ is stretched horizontally between two fixed points. Gravity is neglected. Masses $15m$, $7m$ are attached to the points of trisection. The tension in equilibrium is $\lambda ml$. The particle of mass $15m$ is drawn aside a distance $a$, the other remaining undisplaced, and both are simultaneously released. Prove that in the subsequent motion the displacement of the particle of mass $7m$ is

$$\frac{15}{26}a\{\cos\sqrt{(3\lambda/35)}t-\cos\sqrt{(\lambda/3)}t\}.$$

23. A smooth circular wire, of mass $8m$ and radius $a$, swings in a vertical plane, being suspended by an inextensible string of length $a$ attached to one point of it; a particle of mass $m$ can slide on the wire. Prove that the periods of the normal oscillations are

$$2\pi\sqrt{(8a/3g)},\qquad 2\pi\sqrt{(a/3g)},\qquad 2\pi\sqrt{(8a/9g)}.$$

If the system is released from rest with the ring in its equilibrium position and the mass $m$ displaced through a small angle $\alpha$ from its equilibrium position, show that at time $t$ the angle which the string makes with the vertical is

$$\frac{\alpha}{21}\{\cos t\sqrt{(3g/8a)}-\cos t\sqrt{(3g/a)}\}.$$

24. A light string of length $3l$ is stretched under tension $P$ between two fixed points. Masses $5m$ and $8m$ are attached to it at the points of trisection. The whole is at rest until a small transverse velocity $u$ is suddenly given to the particle of mass $5m$. Prove that in the subsequent motion the displacement of the other particle is

$$\frac{5u}{14\alpha}\left\{\sqrt{\tfrac{20}{3}}\sin\sqrt{\tfrac{3}{20}}\,\alpha t-\sqrt{2}\sin\frac{\alpha t}{\sqrt{2}}\right\},$$

where $\alpha^2 = P/ml$.

25. Two uniform rods $AB$, $CD$, each of length $l$ and mass $4M$, lie on a smooth horizontal table and are freely movable about their ends $A$, $D$, which are fixed at a distance $4l$ apart. $B$, $C$ are joined by an elastic string which carries a particle of mass $M$ at its middle point, the tension of the string being $T$. Initially $B$ is displaced through a small distance $a$ from the position of stable equilibrium while $C$ and the particle $M$ are held in their equilibrium position and the whole system is then released.

Show that the displacement of the particle at any subsequent time is given by

$$y = \tfrac{2}{5}a\{\cos kt\sqrt{\tfrac{2}{3}} - \cos 2kt\},$$

where $k = \sqrt{(3T/4Ml)}$.

26. A light string $OAB$ is tied to a fixed point at $O$, and carries a mass $2m$ at $A$ and a mass $m$ at $B$. The lengths $OA$, $AB$ are $\frac{1}{2}l$, $\frac{3}{4}l$ respectively. The string is free to move in a vertical plane, and the system oscillates about the position of equilibrium. The inclinations of $OA$, $AB$ to the vertical are denoted by $\theta$, $\phi$ respectively. Find the normal coordinates.

The system is held with the string straight and inclined at a small angle $\alpha$ to the vertical, and is let go from rest in this position at the instant $t = 0$. Show that at any subsequent time

$$\theta = \tfrac{1}{3}\alpha(2\cos nt + \cos 2nt),$$
$$\phi = \tfrac{1}{3}\alpha(4\cos nt - \cos 2nt),$$

where $n = \sqrt{(g/l)}$.

27. A light string of length $4a$ is stretched at tension $T$, and particles of masses $m$, $\frac{21}{20}m$, $m$ are attached at the points of quadrisection, with the unequal one in the middle. Find the normal modes for small transverse oscillations.

If the motion be started by a blow $I$ on one of the particles $m$ at the instant $t = 0$, prove that the displacement of the middle particle at any subsequent time is

$$\frac{10}{29}\frac{I}{m}\left(\frac{\sin \alpha t}{\alpha} - \frac{\sin \beta t}{\beta}\right),$$

where $\alpha^2 = \dfrac{4}{7}\dfrac{T}{ma}$, $\beta^2 = \dfrac{10}{3}\dfrac{T}{ma}$.

28. Two equal rods $AB$, $BC$ of length $2a$ resting on a smooth horizontal plane are jointed at $B$ by a spring such that $BC$ can rotate about $B$, the couple required to twist $BC$ till $\angle ABC = \pi - \theta$ being $\lambda\theta$. The rod $AB$ is fastened at $A$ by a similar spring. The system is released from rest in the position in which $AB$ is in its equilibrium position and $BC$ is turned through a small angle $\alpha$ from the direction $AB$. Show that the periods of the principal oscillations are approximately

$$2\pi/0{\cdot}292n \quad \text{and} \quad 2\pi/1{\cdot}942n, \quad \text{where } n^2 = \lambda/ma^2,$$

and that the angular displacement of $AB$ in the subsequent motion is given approximately by

$$0{\cdot}291\alpha n^2\{\cos 0{\cdot}292nt - \cos 1{\cdot}942nt\}.$$

29. A disk $A$ of moment of inertia 100 is attached to a point $B$ by a shaft of stiffness (couple per radian twist) 900 and negligible inertia. The end $B$ of the shaft is given a forced vibration $0{\cdot}1\sin \pi t$, beginning at $t = 0$, when the system is at rest and unstrained. Show that the motion of the disk is given approximately by

$$\theta = 0{\cdot}345\{\pi \sin 3t - 3\sin \pi t\}.$$

30. A disk $C$ of moment of inertia 10 is attached to the disk $A$ of Ex. 29 by a shaft of stiffness 20. Find the natural frequencies of the system and show that the motion of $C$ due to motion $0{\cdot}1 \sin \pi t$ of $B$, beginning at $t = 0$, when the system is at rest and unstrained, is approximately

$$\theta = 0{\cdot}369 \sin \pi t - 0{\cdot}414 \sin 3{\cdot}042t + 0{\cdot}070 \sin 1{\cdot}395t.$$

31. Three flywheels $A$, $B$, $C$, of moments of inertia $3I$, $4I$, $3I$ respectively, are connected by equal shafts $AB$, $BC$ of stiffness $\lambda$ and negligible moment of inertia. At $t = 0$, when the system is at rest and unstrained, $A$ is suddenly given an angular velocity $\omega$. Show that the subsequent angular velocity of $C$ is

$$\frac{\omega}{10}\{3 - 5 \cos nt + 2 \cos nt \sqrt{\tfrac{5}{2}}\},$$

where $n^2 = \lambda/3I$.

## CHAPTER IV

# THE INVERSION THEOREM FOR THE LAPLACE TRANSFORMATION AND ITS APPLICATION TO ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

**28.** In Chapter I § 1, from the differential equation

$$\phi(D)x = F(t), \quad t > 0, \tag{1}$$

with $x_0, x_1, \ldots, x_{n-1}$ for the values of $x, Dx, \ldots, D^{n-1}x$, when $t = 0$, we obtained by means of the Laplace Transformation the subsidiary equation § 1 (5), and for various forms of $F(t)$ we saw that $x(t)$ can be found from this equation with the help of elementary theorems in the differential and integral calculus.

The method adopted was to break up $\bar{x}(p)$ into its partial fractions and for each fraction to write down the function of which it is the Laplace Transform.

An alternative method is to use the Inversion Theorem† for the Laplace Transformation, an integral formula by which $x(t)$ may be obtained from $\bar{x}(p)$. A formal statement of this theorem, without any reference to the conditions to be satisfied by the functions, is that

$$\begin{aligned} &\textit{if} \qquad \bar{x}(p) = \int_0^\infty e^{-pt}x(t)\,dt, \quad \mathrm{R}(p) > 0, \\ &\textit{then}\ddagger \qquad x(t) = \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\bar{x}(\lambda)\,d\lambda, \end{aligned} \tag{2}$$

where $\gamma$ is a constant greater than the real part of all the singularities of $\bar{x}(\lambda)$.

So far we have made no use of this theorem as it seemed better to solve the problems of the introductory chapters by the simplest and most elementary means, but it is essential for the application of the method to partial differential equa-

† It is related to Fourier's and Mellin's integral theorems and is sometimes called the Fourier-Mellin theorem.

‡ By $\int_{\gamma-i\infty}^{\gamma+i\infty}$ we mean $\lim_{w\to\infty}\int_{\gamma-iw}^{\gamma+iw}$.

tions. A complete discussion of the theorem† needs some knowledge of the Theory of Functions of a Complex Variable and is rather difficult. However, if we assume that $x(t)$ satisfies certain fairly general conditions, the formula can be obtained without much trouble. This proof will be given in § 29.‡ As an alternative in § 30 we shall derive the formula from Fourier's Integral Theorem, again imposing certain conditions on $x(t)$.

In § 32 we shall apply the theorem to problems of the type given in the previous chapters. In these applications a slight knowledge of the Theory of Functions of a Complex Variable and of the simplest ideas of the Calculus of Residues will be required.

We shall then return to the question raised in § 5 of the verification of the solutions obtained by either of these methods. It will be remembered that in obtaining the subsidiary equation certain assumptions were made as to the properties of the unknown function $x(t)$, and the Inversion Theorem is also here established with further assumptions regarding it. We shall show that the given differential equation (or system of simultaneous equations) and the conditions imposed when $t = 0$ are satisfied by these solutions, when $F(t)$ in (1), or the corresponding equations, is continuous or has a finite number of ordinary discontinuities, and $\int_0^\infty e^{-pt}F(t)\,dt$ converges absolutely when the real part of $p$ is positive and sufficiently large.

## 29. *The Inversion Theorem.*

*Let $x(t)$ have a continuous derivative, and let $|x(t)| < Ke^{ct}$, where $K$ and $c$ are positive constants. Let*

$$\bar{x}(p) = \int_0^\infty e^{-pt}x(t)\,dt, \qquad \mathrm{R}(p) > c.$$

*Then*
$$x(t) = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t}\bar{x}(\lambda)\,d\lambda, \quad \textit{where } \gamma > c.$$

† Cf. Doetsch, loc. cit., chap. vi.

‡ For this section and § 34 we are indebted to Professor Titchmarsh.

For
$$\frac{1}{2\pi i}\int\limits_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t}\bar{x}(\lambda)\,d\lambda = \frac{1}{2\pi i}\int\limits_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t}\,d\lambda \int\limits_{0}^{\infty} e^{-\lambda u}x(u)\,du$$
$$= \frac{1}{2\pi i}\int\limits_{0}^{\infty} x(u)\,du \int\limits_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda(t-u)}\,d\lambda$$
(since we may invert the order of integration because of the uniform convergence)
$$= \frac{1}{\pi}\int\limits_{0}^{\infty} x(u)e^{\gamma(t-u)}\frac{\sin\omega(t-u)}{t-u}\,du = \frac{1}{\pi}\int\limits_{-t}^{\infty} f(s)\frac{\sin\omega s}{s}\,ds, \tag{1}$$
on putting $u = t+s$ and writing $f(s) = e^{-\gamma s}x(t+s)$.

We break up the integral in (1) into $\int\limits_{0}^{\infty}$ and $\int\limits_{-t}^{0}$. Then the first of these we write
$$\int\limits_{0}^{\infty} f(s)\frac{\sin\omega s}{s}\,ds = f(0)\int\limits_{0}^{\delta}\frac{\sin\omega s}{s}\,ds + \int\limits_{0}^{\delta}\frac{f(s)-f(0)}{s}\sin\omega s\,ds +$$
$$+\int\limits_{\delta}^{\Delta} f(s)\frac{\sin\omega s}{s}\,ds + \int\limits_{\Delta}^{\infty} f(s)\frac{\sin\omega s}{s}\,ds. \tag{2}$$

We can choose $\delta$ and $\Delta$ so that the moduli of the second and fourth integrals in (2) are less than $\epsilon$ for all values of $\omega$. For the third integral we have
$$\int\limits_{\delta}^{\Delta} f(s)\frac{\sin\omega s}{s}\,ds$$
$$= \left[-\frac{\cos\omega s}{\omega s}f(s)\right]_{\delta}^{\Delta} + \frac{1}{\omega}\int\limits_{\delta}^{\Delta}\cos\omega s\,\frac{d}{ds}\left(\frac{1}{s}f(s)\right)ds = \mathbf{O}\left(\frac{1}{\omega}\right).$$

Also for the first integral
$$\int\limits_{0}^{\delta}\frac{\sin\omega s}{s}\,ds = \int\limits_{0}^{\omega\delta}\frac{\sin y}{y}\,dy = \frac{\pi}{2}+\mathbf{O}\left(\frac{1}{\omega}\right).$$

Combining these results we obtain

$$\lim_{\omega\to\infty}\int_0^\infty f(s)\frac{\sin\omega s}{s}\,ds = \tfrac{1}{2}\pi f(0) = \tfrac{1}{2}\pi x(t).$$

Treating the part $\int_{-t}^{0}$ of (1) in the same way we obtain

$$\lim_{\omega\to\infty}\int_{-t}^{0} f(s)\frac{\sin\omega s}{s}\,ds = \tfrac{1}{2}\pi x(t),$$

and the result follows.

**30.** *Deduction of the Inversion Theorem from Fourier's Integral Theorem.*

The usual elementary statement† of Fourier's Integral Theorem is as follows: *Let the arbitrary function $\phi(x)$, defined for all values of $x$, satisfy Dirichlet's conditions‡ in any finite interval, and in addition let $\int_{-\infty}^{\infty}\phi(x)\,dx$ be absolutely convergent. Then*

$$\frac{1}{\pi}\int_0^\infty d\alpha\int_{-\infty}^{\infty}\phi(x')\cos\alpha(x-x')\,dx' = \phi(x) \tag{1}$$

*at every point of continuity and equals $\tfrac{1}{2}[\phi(x+0)+\phi(x-0)]$ at every point where $\phi(x+0)$ and $\phi(x-0)$ exist.*

The repeated integral can be written

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\int_{-\infty}^{\infty}\phi(x')\cos\alpha(x-x')\,dx',$$

and it is clear that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\int_{-\infty}^{\infty}\phi(x')\sin\alpha(x-x')\,dx'$$

† Carslaw, *Fourier Series and Integrals*, 3rd ed. (1930), § 119; Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937), § 1.9.

‡ For a full statement of these see Carslaw, loc. cit. Common types of function which satisfy them are (i) functions with only a finite number of maxima, minima, and ordinary discontinuities; (ii) functions of bounded variation.

is zero. Thus we can replace (1) by

$$\frac{1}{2\pi}\int^{\infty} e^{i\alpha x}\,d\alpha \int^{\infty} \phi(x')e^{-i\alpha x'}\,dx' = \phi(x), \tag{2}$$

for points of continuity, with a corresponding result for points of discontinuity.

Now let

$$\bar{x}(\lambda) = \int_0^{\infty} e^{-\lambda t}x(t)\,dt, \qquad \mathbf{R}(\lambda) \geqslant \gamma > 0.$$

Then

$$\int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t}\bar{x}(\lambda)\,d\lambda = \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda t}\,d\lambda \int_0^{\infty} e^{-\lambda t'}x(t')\,dt'$$

$$= ie^{\gamma t}\int_{-\omega}^{\omega} e^{iyt}\,dy \int_0^{\infty} e^{-iyt'}[e^{-\gamma t'}x(t')]\,dt', \tag{3}$$

on putting $\lambda = \gamma+iy$.

But $$\lim_{\omega\to\infty}\frac{1}{2\pi}\int^{\omega} e^{iyt}\,dy \int^{\infty} e^{-iyt'}[e^{-\gamma t'}x(t')]\,dt'$$

is the Fourier Integral for the function of $t$ equal to $e^{-\gamma t}x(t)$ when $t > 0$ and zero when $t < 0$.

It follows from (3) that

$$x(t) = \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\bar{x}(\lambda)\,d\lambda,$$

provided that $x(t)$ satisfies Dirichlet's conditions in any finite positive interval and $\int_0^{\infty} e^{-\gamma t}x(t)\,dt$ converges absolutely.

**31.** The line integral for $x(t)$ obtained by the use of the Inversion Theorem is usually evaluated by transforming it into a closed contour and applying the calculus of residues. The following simple result permits such a transformation in many cases.

LEMMA. *If* $|f(\lambda)| < CR^{-k}$, *when* $\lambda = Re^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$, $R > R_0$, *where* $R_0$, $C$, $k$ *are constants and* $k > 0$, *then* $\int e^{\lambda t} f(\lambda)\, d\lambda$ *taken over the arcs* $BB'C$ *and* $AA'C$ *of the circle* $\Gamma$ *of radius* $R$ (Fig. 10) *tends to zero as*† $R \to \infty$, *provided* $t > 0$.

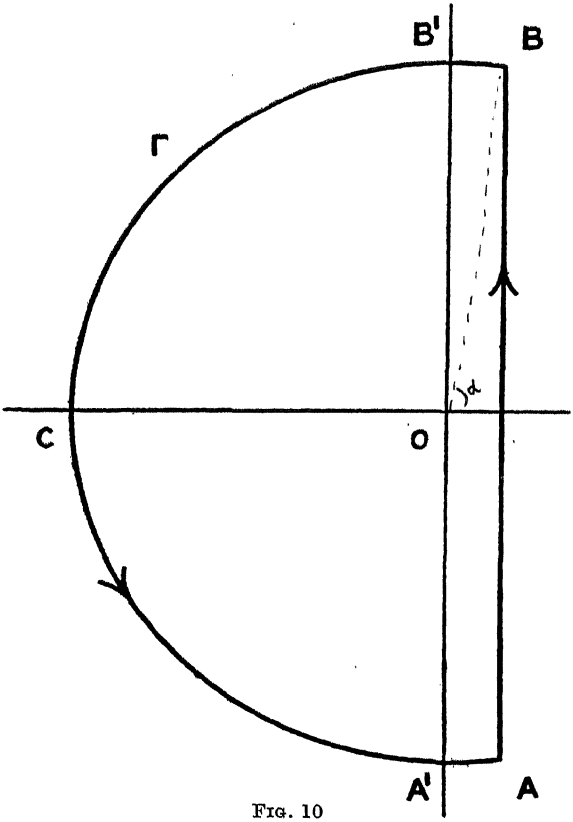


FIG. 10

We consider separately the integrals $I_{BB'}$ and $I_{B'C}$ over the arcs $BB'$ and $B'C$. Those over $AA'$ and $A'C$ are treated in the same way.

† In dealing with partial differential equations it will frequently be necessary to take the radius of the circle $\Gamma$ as a function of the positive integer $n$ and then let $n \to \infty$. Clearly the result holds if $R$ tends to infinity in this way through a sequence of values for which the given conditions are satisfied.

For $BB'$, let $\alpha = \cos^{-1}(\gamma/R)$. Then

$$|I_{BB'}| < CR^{-k+1}e^{\gamma t}\int\limits_{\alpha}^{\frac{1}{2}\pi} d\theta = CR^{-k+1}e^{\gamma t}\sin^{-1}(\gamma/R).$$

Therefore $\lim\limits_{R\to\infty}|I_{BB'}| = 0$.

For† $B'C$,

$$|I_{B'C}| < CR^{-k+1}\int\limits_{\frac{1}{2}\pi}^{\pi} e^{Rt\cos\theta}\,d\theta$$

$$= CR^{-k+1}\int\limits_{0}^{\frac{1}{2}\pi} e^{-Rt\sin\theta}\,d\theta < CR^{-k+1}\int\limits_{0}^{\frac{1}{2}\pi} e^{-2Rt\theta/\pi}\,d\theta < \frac{\pi CR^{-k}}{2t}.$$

Therefore $\lim\limits_{R\to\infty}|I_{B'C}| = 0$ and the result is proved.

In all the problems of this chapter $\bar{x}(\lambda)$ will be a function of $\lambda$ which satisfies the conditions of the lemma and which is analytic except at a finite number of poles all of which are to the left of $\mathrm{R}(\lambda) = \gamma$. It follows that the line integral $(\gamma-i\infty, \gamma+i\infty)$ of the Inversion Theorem may be replaced by the limit of the integral over the closed contour of Fig. 10, when $R\to\infty$, and this in turn may be replaced by any circle $C$, centre the origin, which includes all the poles of $\bar{x}(\lambda)$.

**32.** *Examples of ordinary linear differential equations with constant coefficients solved by the use of the Inversion Theorem.*

Ex. 1. $\quad (D^2+4D+4)x = \sin\omega t, \quad t > 0,$

with $x_0$ and $x_1$ for the values of $x$ and $Dx$ when $t = 0$.

The subsidiary equation is

$$(p+2)^2\bar{x} = (px_0+x_1)+4x_0+\frac{\omega}{p^2+\omega^2}.$$

Thus $$\bar{x} = \frac{px_0+x_1+4x_0}{(p+2)^2}+\frac{\omega}{(p^2+\omega^2)(p+2)^2}.$$

It follows from the Inversion Theorem that

$$x(t) = \frac{1}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\lambda x_0+x_1+4x_0}{(\lambda+2)^2}\,d\lambda+\frac{\omega}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{d\lambda}{(\lambda^2+\omega^2)(\lambda+2)} \tag{1}$$

where $\gamma > 0$.

† In the reduction we use the result $1 > \frac{\sin\theta}{\theta} \geqslant \frac{2}{\pi}$ if $0 < \ \leqslant \frac{\ddot{}}{2}$.

Since the multipliers of $e^{\lambda t}$ in the integrands of (1) are all $\mathbf{O}(\lambda^{-k})$, $k > 0$, it follows from the lemma of § 31 that, if $t > 0$, the line integrals may be replaced by the integral round the contour of Fig. 10, and this in turn may be replaced by the integral round any circle $C$, centre the origin, enclosing the singularities $\pm i\omega$, and $-2$ of the integrand.

The Calculus of Residues then gives us at once the terms in $x(t)$ corresponding to each of the integrals in (1).

For the first we have

$$\left[\frac{d}{d\lambda}e^{\lambda t}(\lambda x_0+x_1+4x_0)\right]_{\lambda=-2},$$

i.e.

$$[x_0+(x_1+2x_0)t]e^{-2t}.$$

In the second,

the pole at $i\omega$ gives $\quad -\dfrac{i(2-i\omega)^2}{2(\omega^2+4)^2}e^{i\omega t}$

and the pole at $-i\omega$ gives $\quad \dfrac{i(2+i\omega)^2}{2(\omega^2+4)^2}e^{-i\omega t}.$

Adding these, we have

$$\frac{(4-\omega^2)\sin\omega t-4\omega\cos\omega t}{(\omega^2+4)^2}$$

The pole at $-2$ gives

$$\omega\left[\frac{d}{d\lambda}\left(\frac{e^{\lambda t}}{(\lambda^2+\omega^2)}\right)\right]_{\lambda=-2}=\frac{\omega e^{-2t}}{\omega^2+4}\left(t+\frac{}{\omega^2+4}\right).$$

Thus

$$x(t) = \{x_0+(x_1+2x_0)t\}e^{-2t}+$$
$$+\frac{(4-\omega^2)\sin\omega t-4\omega\cos\omega t}{(\omega^2+4)^2}+\frac{\omega e^{-2t}}{\omega^2+4}\left(t+\frac{4}{\omega^2+4}\right).$$

In the method used in the earlier chapters we would have proceeded as follows:

$$\bar{x}(p) \quad \frac{x_0}{p+2}+\frac{x_1+2x_0}{(p+2)^2}+\frac{}{(p^2+\omega^2)(p+2)^2}$$
$$=\frac{x_0}{p+2}+\frac{x_1+2x_0}{(p+2)^2}+\frac{\omega}{\omega^2+4}\left\{\frac{1}{(p+2)^2}+\frac{4}{(\omega^2+4)(p+2)}+\frac{4-\omega^2-4p}{(\omega^2+4)(p^2+\omega^2)}\right\}.$$

Therefore

$$x(t) = x_0 e^{-2t}+(x_1+2x_0)te^{-2t}+$$
$$+\frac{\omega}{\omega^2+4}\left[te^{-2t}+\frac{4}{\omega^2+4}e^{-2t}+\frac{1}{\omega(\omega^2+4)}\{(4-\omega^2)\sin\omega t-4\omega\cos\omega t\}\right].$$

Ex. 2. The problem of § 6, Ex. 4.

$$(D^2-2D+2)(D^2+2D-3)x = 0, \quad t > 0,$$

with $x$, $Dx$, $D^2x$, $D^3x$ equal to 1, 0, 6, $-14$ when $t = 0$.

As before, we find

$$\bar{x} = \frac{p^3+p-4}{(p^2-2p+2)(p^2+2p-3)}.$$

The Inversion Theorem gives

$$x = \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}(\lambda^3+\lambda-4)}{(\lambda^2-2\lambda+2)(\lambda^2+2\lambda-3)}\,d\lambda.$$

The path $(\gamma-i\infty, \gamma+i\infty)$ may be replaced by a circle $C$ containing all the poles $-3$, 1, $1\pm i$ of the integrand, and $x$ is equal to $2\pi i$ times the sum of the residues at these poles.

The pole at $\lambda = -3$ gives $\frac{1}{2}e^{-3t}$;

that at $\lambda = 1$ gives $-\frac{1}{2}e^{t}$;

those at $\lambda = 1\pm i$ give $\frac{1}{2}(1\mp i)e^{(1\pm i)t}$.

Adding these we obtain

$$x = e^{t}(\cos t+\sin t)-\tfrac{1}{2}e^{t}+\tfrac{1}{2}e^{-3t}.$$

Ex. 3. $\qquad (D^2+a^2)^2x = \cos at.$

$x$, $Dx$, $D^2x$, $D^3x$ zero when $t = 0$.

Here $$\bar{x} = \frac{p}{(p^2+a^2)^3}.$$

Therefore $$= \frac{1}{2i\pi}\int e^{\lambda t}\frac{\lambda}{(\lambda^2+a^2)^3}\,d\lambda,$$

where the integral is taken round any circle $C$ of radius greater than $a$.

Now $$\left[\frac{d^2}{d\lambda^2}\left(e^{\lambda t}\frac{\lambda}{(\lambda+ia)^3}\right)\right]_{\lambda=ia} = -\frac{t}{8a^2}\left(t+\frac{i}{a}\right)e^{iat}.$$

Thus the pole at $\lambda = ia$ gives $-\dfrac{t}{16a^2}\left(t+\dfrac{i}{a}\right)e^{iat}$

Also the pole at $\lambda = -ia$ gives $-\dfrac{t}{16a^2}\left(t-\dfrac{i}{a}\right)e^{-iat}$

Adding these we have

$$x = \frac{t}{8a^3}[\sin at - at\cos at].$$

**33.** We now prove Theorem VI of Chapter I, § 3, which was stated there without proof.

*Let*
$$f_1(p) = \int_0^\infty e^{-pu} F_1(u)\, du \tag{1}$$

*and*
$$f_2(p) = \int_0^\infty e^{-pv} F_2(v)\, dv \tag{2}$$

*converge absolutely for* $p = p_0 > 0$.

*Then*
$$f_1(p) f_2(p) = \int_0^\infty e^{-pt} F(t)\, dt, \tag{3}$$

*where*
$$F(t) = \int_0^t F_1(t-\tau) F_2(\tau)\, d\tau = \int_0^t F_1(\tau) F_2(t-\tau)\, d\tau, \tag{4}$$

*for* $p \geqslant p_0$.

On account of the absolute convergence of the integrals (1) and (2) we know that

$$\int_0^\infty e^{-p_0 u} F_1(u)\, du \int_0^\infty e^{-p_0 v} F_2(v)\, dv = \iint e^{-p_0(u+v)} F_1(u) F_2(v)\, du dv,$$

where the double integral is taken over the quadrant $u > 0$, $v > 0$.

The substitution $u+v = t$ and $v = \tau$ transforms this double integral into

$$\iint e^{-p_0 t} F_1(t-\tau) F_2(\tau)\, dt d\tau,$$

taken over the region between the $t$-axis and the line $t = \tau$ in the $(t, \tau)$ plane.

This double integral is equal to the repeated integral

$$\int_0^\infty e^{-p_0 t}\left[\int_0^t F_1(t-\tau) F_2(\tau)\, d\tau\right] dt.$$

Since the absolute convergence of (1) and (2) for $p = p_0 > 0$ clearly carries with it their absolute convergence for $p > p_0$, the theorem as stated is proved.

**34.** *Verification of the solution obtained by the Laplace Transformation method for*

$$\phi(D)x = F(t),\dagger$$

*with* $x_0, x_1,\ldots, x_{n-1}$ *for* $x, Dx,\ldots, D^{n-1}x$, *when* $t = 0$.

Write the subsidiary equation of § 1 (5) in the form

$$\phi(p)\int_0^\infty e^{-pt}x\,dt = \chi(p)+\bar{F}(p),$$

where

$$\begin{aligned}\chi(p) = {} & x_0(p^{n-1}+a_1 p^{n-2}+\ldots+a_{n-1})+\\ & +x_1(p^{n-2}+a_1 p^{n-3}+\ldots+a_{n-2})+\\ & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ & +x_{n-2}(p+a_1)+\\ & +x_{n-1}\end{aligned}$$

and

$$\bar{F}(p) = \int^\infty e^{-pt}F(t)\,dt.$$

The solution is given by

$$x(t) = x_1(t)+x_2(t), \tag{1}$$

where

$$x_1(t) = \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\chi(\lambda)}{\phi(\lambda)}\,d\lambda, \tag{2}$$

and, by Theorem VI, § 33,

$$x_2(t) = \int_0^t Q(t-\tau)F(\tau)\,d\tau, \tag{3}$$

$Q(t)$ being given by‡

$$\frac{1}{\phi(p)} = \int^\infty e^{-pt}Q(t)\,dt,$$

or

$$Q(t) = \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{d\lambda}{\phi(\lambda)}. \tag{4}$$

† $\int_0^\infty e^{-pt}F(t)\,dt$ is supposed absolutely convergent for $p$ with a real part sufficiently large and positive.

‡ It is clear that the absolute convergence required in Theorem VI as proved in § 33 is satisfied by $\int^\infty e^{-pt}Q(t)\,dt$.

Since $\chi(\lambda)$ and $\phi(\lambda)$ are polynomials in $\lambda$, the degree of the first being at least one lower than that of the second, it follows that the integrands of (2) and (4) satisfy the conditions of the lemma of §31, and thus that, if $t > 0$, the line integrals $(\gamma - i\infty, \gamma + i\infty)$ in (2) and (4) may be replaced by integrals over a circle $C$, centre the origin, which includes all the zeros of $\phi(\lambda)$.

By expanding in ascending powers of $1/p$ it will be seen that

$$\frac{\chi(p)}{\phi(p)} = \frac{x_0}{p} + \frac{x_1}{p^2} + \ldots + \frac{x_{n-1}}{p^n} + O\left(\frac{1}{p^{n+1}}\right). \tag{5}$$

Thus

$$x_1^{(m)}(t)$$

$$= \frac{1}{2i\pi} \int\limits_C e^{\lambda t} \lambda^m \left(\frac{x_0}{\lambda} + \frac{x_1}{\lambda^2} + \ldots + \frac{x_{n-1}}{\lambda^n}\right) d\lambda + \frac{1}{2i\pi} \int\limits_C e^{\lambda t} O\left(\frac{1}{\lambda^{n+1-m}}\right) d\lambda.$$

It follows that $\lim\limits_{t \to 0} x_1^m(t) = x_m$, $m = 0, 1, \ldots, n-1$, since when $t = 0$ the last term tends to zero as the radius of the circle $C$ tends to infinity.†

Also it follows from (4) that

$$Q^{(m)}(t) = \frac{1}{2i\pi} \int\limits_C e^{\lambda t} \frac{\lambda^m \, d\lambda}{\phi(\lambda)}, \tag{6}$$

and thus
$$\lim_{t \to 0} Q^{(m)}(t) = 0, \quad \text{when } m \leqslant n-2,$$
$$= 1, \quad \text{when } m = n-1.$$

Now consider
$$x_2(t) = \int\limits_0^t Q(t-\tau) F(\tau) \, d\tau.$$

Then, for values of $t$ at which $F(t)$ is continuous,

$$x_2'(t) = Q(0)F(t) + \int\limits_0^t Q'(t-\tau)F(\tau) \, d\tau = \int\limits_0^t Q'(t-\tau)F(\tau) \, d\tau,$$

† Since $\int\limits_C \lambda^m e^{\lambda t} \frac{\chi(\lambda)}{\phi(\lambda)} \, d\lambda$ is continuous for $t \geqslant 0$,

$$\lim_{t \to 0} x_1^{(m)}(t) = \int\limits_C \lambda^m \frac{\chi(\lambda)}{\phi(\lambda)} \, d\lambda.$$

and

$$x_2''(t) = Q'(0)F(t) + \int_0^t Q''(t-\tau)F(\tau)\,d\tau = \int_0^t Q''(t-\tau)F(\tau)\,d\tau,$$

. . . . . . . . . . . . . . . . .

$$x_2^{(n-1)}(t) \qquad\qquad = \int_0^t Q^{(n-1)}(t-\tau)F(\tau)\,d\tau.$$

But
$$x_2^{(n)}(t) = F(t) + \int_0^t Q^{(n)}(t-\tau)F(\tau)\,d\tau.$$

Thus
$$\phi(D)x_2(t) = F(t) + \int_0^t F(\tau)\phi(D)Q(t-\tau)\,d\tau$$
$$= F(t), \tag{7}$$

since, by adding results of type (6),

$$\phi(D)Q(t) = \frac{1}{2i\pi}\int_C e^{\lambda t}\,d\lambda = 0.$$

Also
$$\phi(D)x_1(t) = \frac{1}{2i\pi}\int_C \chi(\lambda)e^{\lambda t}\,d\lambda = 0. \tag{8}$$

Thus, from (7) and (8),

$$\phi(D)x(t) = F(t),$$

and the verification is complete.

**35.** For simultaneous equations we omit the general case and consider only the system of $n$ first-order† equations

$$\sum_{s=1}^{n} e_{rs}x_s = F_r(t), \quad r = 1, 2,..., n, \quad t > 0,$$

where
$$e_{rs} = a_{rs}D + b_{rs}, \tag{1}$$

and $x_r = u_r$, $r = 1,..., n$, when $t = 0$.‡

The problem is simplified by breaking it up into two:

I. The non-homogeneous case:

$$\sum_{s=1}^{n} e_{rs}x_s = F_r(t), \quad r = 1,..., n, \quad t > 0,$$

with $x_r = 0$, $r = 1, 2,..., n$, when $t = 0$.

† A system of higher order can be reduced to a first-order system by substitution.

‡ The footnote on p. 81 also applies to the functions $F_r(t)$.

II. The homogeneous case:

$$\sum_{s=1}^{n} e_{rs} x_s = 0, \quad r = 1,\ldots, n, \quad t > 0,$$

with $x_r = u_r$, $r = 1,\ldots, n$, when $t = 0$.

I. *The non-homogeneous case*. This can be further simplified by taking all but one of the $F$'s zero and adding the solutions thus obtained. Thus we take the system of equations

$$\left.\begin{aligned} \sum^{n} e_{1s} x_s &= F(t), \\ \sum_{s=1}^{n} e_{rs} x_s &= 0, \quad r = 2, 3,\ldots, n, \end{aligned}\right\} \quad t > 0, \tag{2}$$

with $x_r = 0$, $r = 1,\ldots, n$, when $t = 0$.

The subsidiary equations are

$$\begin{aligned} \sum_{s=1} p_{1s} \bar{x}_s &= \bar{F}(p), \\ \sum_{s=1}^{n} p_{rs} \bar{x}_s &= 0, \quad r = 2, 3,\ldots, n, \end{aligned} \tag{3}$$

where
$$p_{rs} = a_{rs} p + b_{rs}. \tag{4}$$

Thus
$$\bar{x}_r = P_{1r} \bar{F}(p)/\Delta,$$

where
$$\Delta = \begin{vmatrix} p_{11} & p_{12} & . & . & p_{1n} \\ p_{21} & . & . & . & p_{2n} \\ . & . & . & . & . \\ p_{n1} & . & . & . & p_{nn} \end{vmatrix} \tag{5}$$

and $P_{rs}$ is the cofactor of $p_{rs}$ in this determinant.

Let
$$\frac{P_{rs}}{\Delta} = \int_0^\infty e^{-pt} Q_{rs}(t)\, dt,$$

so that
$$Q_{rs}(t) = \frac{1}{2i\pi} \int_{\mathcal{L}} e^{\lambda t} \frac{P_{rs}(\lambda)}{\Delta(\lambda)}\, d\lambda. \tag{6}$$

Then the solution of (2) with its initial conditions is

$$x_r = \int_0^t Q_{1r}(t-\tau) F(\tau)\, d\tau, \quad r = 1, 2,\ldots, n. \tag{7}$$

*Verification.* We have

$$Q_{rs} = \frac{1}{2i\pi}\int e^{\lambda t}\frac{P_{rs}(\lambda)}{\Delta(\lambda)}\,d\lambda = \frac{1}{2i\pi}\int e^{\lambda t}\frac{A_{rs}\lambda^{n-1}+\dots}{A\lambda^{n}+\dots}\,d\lambda,$$

where

$$A = \begin{vmatrix} a_{11} & a_{12} & & \\ & & & \\ a_{n1} & a_{n2} & \cdot\;\;\cdot & a_{nn} \end{vmatrix} \tag{8}$$

and $A_{rs}$ is the cofactor of $a_{rs}$ in the expansion of this determinant. We assume $A \neq 0$. Then

$$Q_{rs}(0) = \frac{1}{2i\pi}\int\limits_{C}\left[\frac{A_{rs}}{A\lambda}+O\left(\frac{1}{\lambda^2}\right)\right]d\lambda = \frac{A_{rs}}{A}. \tag{9}$$

Now
$$Dx_s = Q_{1s}(0)F(t)+\int\limits_0^t Q'_{1s}(t-\tau)F(\tau)\,d\tau.$$

Therefore
$$e_{1s}x_s = \frac{a_{1s}A_{1s}}{A}F(t)+\int\limits_0^t F(\tau)e_{1s}Q_{1s}(t-\tau)\,d\tau.$$

Thus, adding,

$$\sum_{s=1}^{n} e_{1s}x_s = F(t)+\int\limits_0^t F(\tau)\Big\{\sum_{s=1}^{n} e_{1s}Q_{1s}(t-\tau)\Big\}\,d\tau$$
$$= F(t),$$

since
$$\sum_{s=1}^{n} e_{ms}Q_{rs} = \frac{1}{2i\pi}\sum_{s=1}^{n}\int\limits_{C} e^{\lambda t}\frac{p_{ms}(\lambda)P_{rs}(\lambda)}{\Delta(\lambda)}\,d\lambda$$
$$= 0, \quad \text{if } r \neq m,$$
$$= \frac{1}{2i\pi}\int e^{\lambda t}\,d\lambda = 0, \quad r = m. \tag{10}$$

Hence the first equation is satisfied, and the others are treated similarly.

II. *The homogeneous case.* The equations are

$$\sum_{s=1}^{n} e_{rs}x_s = 0, \quad r = 1,\dots, n, \quad t > 0,$$

with $x_r = u_r$, $r = 1,\dots, n$, when $t = 0$.

The subsidiary equations are

$$\sum_{s=1}^{n} p_{rs}\bar{x}_s = U_r, \quad r = 1,.$$

where

$$U_r = \sum_{s=1}^{n} a_{rs} u_s. \tag{11}$$

Thus

$$\bar{x}_s = \frac{1}{\Delta}\sum_{r=1}^{n} P_{rs} U_r,$$

and

$$x_s = \sum_{r=1}^{n} Q_{rs} U_r,$$

where $P_{rs}$ and $Q_{rs}$ are defined in (5) and (6).

*Verification.* It follows from (10) that

$$\sum_{s=1}^{n} e_{rs} x_s = \sum_{m=1}^{n} U_m \sum_{s=1}^{n} e_{rs} Q_{ms} = 0, \quad r = 1,\ldots, n.$$

Also, when $t = 0$,

$$x_s = \sum_{r=1}^{n} U_r Q_{rs}(0) = \frac{1}{A}\sum_{r=1}^{n} U_r A_{rs}$$

$$= \frac{1}{A}\sum_{m=1}^{n} u_m \sum_{r=1}^{n} a_{rm} A_{rs}$$

$$= u_s, \quad s = 1, 2,\ldots, n.$$

## EXAMPLES ON CHAPTER IV

Use the Inversion Formula to solve the equations in Exs. 1–10, with the given conditions when $t = 0$. The answer is given at the end of each question. $D$ is used for $d/dt$.

1. $D(D-1)x = t^2, \quad x_0, x_1.$

$$\left[(x_0-x_1)+x_1 e^t - 2\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}-e^t\right).\right.$$

2. $(D^3+1)x = \frac{1}{2}t^2e^t, \quad x_0 = -x_1 = x_2.$

$$\left[(x_0-\tfrac{1}{24})e^{-t}+\tfrac{1}{4}(t^2-3t+\tfrac{3}{2})e^t-\tfrac{2}{3}e^{\frac{1}{2}t}\cos(\tfrac{1}{3}\pi+\tfrac{1}{2}\sqrt{3}t).\right.$$

3. $(D^2+1)x = t\cos 2t, \quad x_0, x_1.$

$$[x_0\cos t+x_1\sin t-\tfrac{5}{9}\sin t-\tfrac{1}{3}t\cos 2t+\tfrac{4}{9}\sin 2t.$$

4. $(D^2-3D+2)x = e^t, \quad x_0, x_1.$

$$[(x_1-x_0+1)e^{2t}+(2x_0-x_1-1-t)e^t.$$

5. $(D^2-2D+2)(D^2-2D-3)x = \sin t, \quad x_0, x_1, x_2$, and $x_3$ all zero.

$$[\tfrac{21}{200}e^{3t}+\tfrac{11}{40}e^{-t}+\tfrac{1}{25}(17\cos t-16\sin t)e^t-\tfrac{1}{50}(3\cos t+4\sin t).$$

6. $(D^2+m^2)^2x = a\sin mt$, $x_0$, $x_1$, $x_2$, and $x_3$ all zero.

$$\left[\frac{a}{8m^2}\left\{\left(\frac{3}{m^2}-t^2\right)\sin mt-\frac{3t}{m}\cos mt\right\}.\right.$$

7. $(D^2+m^2)^2x = a\sin nt$, $x_0$, $x_1$, $x_2$, and $x_3$ all zero.

$$\left[\frac{a}{(m^2-n^2)^2}\sin nt+\frac{an}{2m^2(m^2-n^2)}\left\{t\cos mt-\frac{3m^2-n^2}{m(m^2-n^2)}\sin mt\right\}.\right.$$

8. $(D^2+m^2)^2x = a\sin(mt+\alpha)$, $x_0$, $x_1$, $x_2$, and $x_3$ all zero.

[See p. 22, Ex. 17.

9. $(D^2+m^2)^2x = a\sin(nt+\alpha)$, $x_0$, $x_1$, $x_2$, and $x_3$ all zero.

[See p. 22, Ex. 18.

10. $(D-1)^nx = 1$, $x_0, x_1,\ldots, x_{n-1}$ all zero.

$$\left[(-1)^n+(-1)^{n-1}\left\{1-t+\frac{t^2}{2!}-\ldots+(-1)^{n-1}\frac{t^{n-1}}{(n-1)!}\right\}e^t.\right.$$

11. From the integral

$$\int_0^\infty e^{-a^2x^2-b^2/x^2}\,dx = \frac{\sqrt\pi}{2a}e^{-2ab}, \quad a \text{ and } b \text{ positive},$$

show that, if $q = \sqrt{(p/\kappa)}$,

(i) $\dfrac{e^{-qx}}{\kappa q}$ is the Laplace Transform of $\dfrac{e^{-x^2/4\kappa t}}{\sqrt{(\pi\kappa t)}}$, $x > 0$,

(ii) $e^{-qx}$ is the Laplace Transform of $\dfrac{x}{2\sqrt{(\pi\kappa t^3)}}e^{-x^2/4\kappa t}$, $x > 0$;

and from (ii) deduce that

$\dfrac{e^{-qx}}{p}$ is the Laplace Transform of $1-\operatorname{erf}\{x/2\sqrt{(\kappa t)}\}$.

12. Use the Inversion Formula to obtain $x_1(t)$ and $x_2(t)$, when

(i) $\bar{x}_1(p) = \dfrac{p^2-a^2}{(p^2+a^2)^2}$. [$t\cos at$.

(ii) $\bar{x}_2(p) = \dfrac{2ap}{(p^2+a^2)^2}$. [$t\sin at$.

13. Show that $$\frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{d\lambda}{(\lambda-ia)^{n+1}} = \frac{t^ne^{iat}}{n!},$$

and deduce that

$$\int_0^\infty e^{-pt}t^n\cos at\,dt = n!\times\text{real part of }\frac{1}{(p-ia)^{n+1}},$$

$$\int_0^\infty e^{-pt}t^n\sin at\,dt = n!\times\text{imaginary part of }\frac{1}{(p-ia)^{n+1}}.$$

## CHAPTER V

# LAPLACE TRANSFORM METHOD IN THE SOLUTION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

**36.** A similar method can be used in the solution of partial differential equations such as occur in various branches of applied mathematics, with initial and boundary conditions.

For example, take the equation

$$\nabla^2 u + A_2(x,y,z)\frac{\partial^2 u}{\partial t^2} + A_1(x,y,z)\frac{\partial u}{\partial t} + A_0(x,y,z)u = B(x,y,z,t), \tag{1}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$, $(x,y,z)$ is a point in a given region, and $t$, usually the time, is positive.

A boundary condition is to be satisfied of the form

$$G(x,y,z)u + H(x,y,z)\frac{\partial u}{\partial n} = K(x,y,z,t), \tag{2}$$

where $\partial/\partial n$ denotes differentiation along the normal.

There are also conditions for $t = 0$ within the region; e.g.

$$\left.\begin{aligned} \lim_{t\to 0} u(x,y,z,t) &= u_0(x,y,z), \\ \lim_{t\to 0} \frac{\partial}{\partial t} u(x,y,z,t) &= u_1(x,y,z). \end{aligned}\right\} \tag{3}$$

We multiply (1) by $e^{-pt}$ $(p > 0)$ and integrate with respect to $t$ from 0 to $\infty$, assuming that

$$\int_0^\infty e^{-pt}u\,dt, \qquad \int_0^\infty e^{-pt}\frac{\partial u}{\partial t}\,dt, \quad \text{etc.},$$

exist. Also we assume that

$$\int_0^\infty e^{-pt}\nabla^2 u\,dt = \nabla^2 \int e^{-pt}u\,dt.$$

But, as in § 1,

$$\int_0^\infty e^{-pt}\frac{\partial u}{\partial t}\,dt = [e^{-pt}u]_0^\infty + p\int_0^\infty e^{-pt}u\,dt$$

$$= -u_0 + p\bar{u},$$

where $$\bar{u} = \int_0^\infty e^{-pt}u\,dt.$$

Also $$\int_0^\infty e^{-pt}\frac{\partial^2 u}{\partial t^2}\,dt = \left[e^{-pt}\frac{\partial u}{\partial t}\right]_0^\infty + p\int_0^\infty e^{-pt}\frac{\partial u}{\partial t}\,dt$$
$$= -u_1 + p\int_0^\infty e^{-pt}\frac{\partial u}{\partial t}\,dt$$
$$= -(pu_0+u_1)+p^2\bar{u}.$$

Thus, with the above assumptions as to the nature of the unknown function $u$, we obtain from (1) and (3) the 'subsidiary equation'

$$\nabla^2\bar{u}+[A_2(x,y,z)p^2+A_1(x,y,z)p+A_0(x,y,z)]\bar{u}$$
$$= A_2(x,y,z)[pu_0+u_1]+A_1(x,y,z)u_0+\int_0^\infty e^{-pt}B(x,y,z,t)\,dt. \quad (4)$$

The boundary condition (2) is replaced by

$$G(x,y,z)\bar{u}+H(x,y,z)\frac{\partial \bar{u}}{\partial n} = \int_0^\infty e^{-pt}K(x,y,z,t)\,dt. \quad (5)$$

**37.** If we can find $\bar{u}$ from (4) and (5) of § 36, our problem is reduced to finding $u$ from the equation

$$\bar{u}(x,y,z,p) = \int_0^\infty e^{-pt}u(x,y,z,t)\,dt.$$

It may happen that $\bar{u}(x,y,z,p)$ appears in the Table of Transforms and then $u(x,y,z,t)$ can be written down directly. If this is not the case, we obtain $u$ from $\bar{u}$ by the Inversion Theorem, § 29, namely,

$$u(x,y,z,t) = \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\bar{u}(x,y,z,\lambda)\,d\lambda. \quad (1)$$

The line integral is then evaluated by transformation to a suitable closed contour and the use of the Calculus of Residues. Most of the problems with which we shall be concerned fall into two classes according to the nature of $\bar{u}(x,y,z,\lambda)$ as a function of $\lambda$:

I. $\bar{u}(x, y, z, \lambda)$ is a single-valued function of $\lambda$ with an enumerably infinite number of poles. In this case we use the contour of Fig. 10, choosing for the radius $R$ of the large circle $\Gamma$ a sequence of values $R_n$ such that none of these circles passes through any pole of $\bar{u}(x, y, z, \lambda)$. Then, if $\bar{u}(x, y, z, \lambda)$ satisfies the conditions of the lemma of § 31, the integral round the circle of radius $R_n$ in Fig. 10 tends to zero as $n \to \infty$. Thus, by Cauchy's theorem, the line integral in (1) may be replaced by the limit as $n \to \infty$ of $2\pi i$ times the sum of the residues of the integrand at its poles within the circle $\Gamma$ of radius $R_n$.

II. $\bar{u}(\lambda)$ has a branch point at the origin but otherwise only a finite number of poles. In this case we use the contour of Fig. 11, § 39. If $\bar{u}(x, y, z, \lambda)$ satisfies the conditions of the lemma of § 31, the integrals round the arcs $BF$ and $AC$ of the large circle $\Gamma$ tend to zero as its radius $R$ tends to infinity. The integrand will be single-valued within and on the contour so that, when $R \to \infty$, the solution is obtained as a real infinite integral (derived from the integrals along $CD$ and $EF$) together with $2\pi i$ times the sum of the residues at poles within the contour, and possibly a term arising from the integral round the small circle about the origin.

Other types, such as those of Chapter IX which have two branch points, will be dealt with as they arise.

In §§ 39–44 three simple examples are solved to illustrate the methods of procedure. The proofs that the integrals round the large circle $\Gamma$ of radius $R$ tend to zero as $R \to \infty$ are given in detail in these examples. In the problems of later chapters they will be omitted, but in all cases they can be supplied along the lines of those in §§ 39, 41, 43.

**38.** In deriving the subsidiary equation and its boundary conditions in § 36, and in obtaining $u(x, y, z, t)$ from $\bar{u}(x, y, z, p)$ by the use of the Inversion Theorem, certain assumptions as to the nature of $u(x, y, z, t)$ were made, so that the previous discussion is incomplete. The same difficulty arose for the ordinary differential equation, but there it was found possible to give a general verification process, §§ 34, 35, which would cover all

cases. For partial differential equations this is not possible and a separate treatment must be given for each problem.

It should be remarked that the assumptions referred to above are reasonable and not at all restrictive from a physical point of view. If we assume *a priori* that there exists a solution of the problem with these properties the preceding analysis becomes valid. This point of view seems adequate for most purposes in applied mathematics.

To make the solution completely rigorous it is necessary to verify that the result obtained does satisfy the original differential equation and the initial and boundary conditions. This must be done for the particular problem under consideration. The final form of the solution may be best to work upon. In other cases it may be better to deal with the line integral $\int_{\gamma-i\infty}^{\gamma+i\infty}$ obtained from the Inversion Theorem, and there it may be possible and advisable to change the path $L$ into the path $L'$ of Fig. 15. We shall return to this question of verification in § 58; for the present we take the point of view of the preceding paragraph and simply remark here that for complete rigour verification is necessary and that it can be performed for most of the solutions obtained in the text.†

**39.** *Linear flow of heat in a semi-infinite solid, $x > 0$: the boundary $x = 0$ kept at a constant temperature $v_0$; the initial temperature of the solid zero.*

Here we have to solve

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}, \quad x > 0,\ t > 0, \tag{1}$$

with

$$v = v_0, \quad \text{when } x = 0,\ t > 0, \tag{2}$$

$$v = 0, \quad \text{when } x > 0,\ t = 0. \tag{3}$$

† An entirely different method has been used by Churchill (*Math. Zeitschrift*, **42** (1937), 567, and **43** (1938), 743; *Math. Ann.* **114** (1937), 591, and **115** (1938), 720). The Inversion Theorem, § 29, was proved under conditions on $x(t)$. In place of this it might have been proved under conditions on $\bar{x}(p)$. Further, it is possible to state conditions on $\bar{x}(p)$ under which $x(t)$ obtained from $\bar{x}(p)$ by the Inversion Theorem has the properties required for the verification.

The subsidiary equation is

$$\frac{d^2\bar{v}}{dx^2}-\frac{p}{\kappa}\bar{v}=0, \quad x>0, \tag{4}$$

with
$$\bar{v}=\frac{v_0}{p}, \quad \text{when } x=0. \tag{5}$$

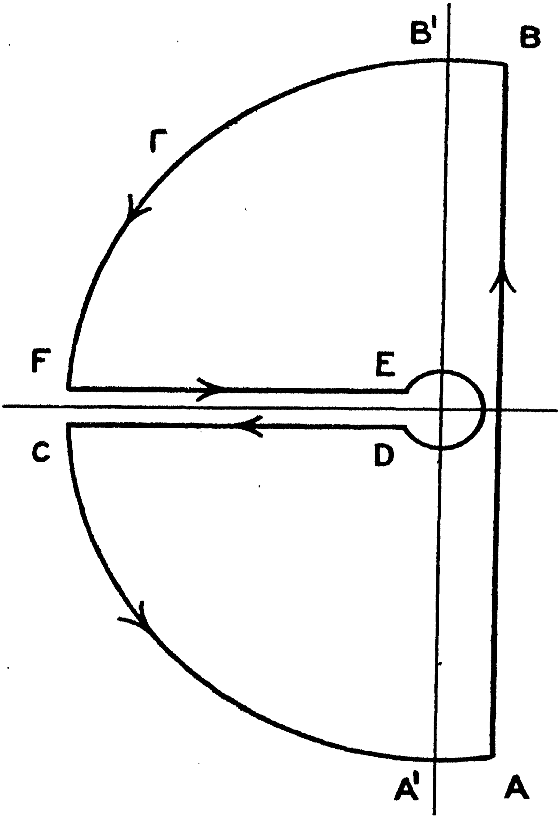


FIG. 11

The solution of (4) and (5) which remains finite when $x \to \infty$ is

$$\bar{v}=\frac{v_0}{p}e^{-x\sqrt{(p/\kappa)}}. \tag{6}$$

Then, using the Inversion Theorem, we have

$$v = \frac{v_0}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t - x\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda}. \tag{7}$$

The integrand of (7) has a branch-point at the origin, so in evaluating (7) by contour integration we have to choose a path which does not contain the origin.

Consider the closed circuit of Fig. 11, where $AB$ is parallel to the imaginary axis at a distance $\gamma$ from it. The circle $\Gamma$ of radius $R$ and centre at the origin cuts this line at $A$ and $B$ and the imaginary axis at $A'$ and $B'$. There is a 'cut' along the negative real axis. The small circle with its centre at 0 is of radius $\epsilon$. This circle is open at $DE$ and $\Gamma$ is open at $CF$. The argument of $\lambda$ is $-\pi$ on $CD$ and $\pi$ on $EF$.

We take
$$\frac{1}{2i\pi}\int e^{\lambda t - x\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda}$$

over this closed circuit and we know that the integral is zero.

Since $\frac{1}{\lambda}e^{-x\sqrt{(\lambda/\kappa)}} < |\lambda|^{-1}$ on $BF$ and $AC$ the conditions of the lemma of § 31 are satisfied, and it follows that the integrals over $BF$ and $AC$ tend to zero as $R \to \infty$.

It follows that

$$\frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t - x\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda}$$

is equal to the sum of the integrals over $CD$ and $EF$, and that over the small circle, when $R \to \infty$ and $\epsilon \to 0$. In the integrals along $CD$ and $EF$ put $\lambda = \rho e^{-i\pi}$ and $\lambda = \rho e^{i\pi}$ respectively, and we get

$$-\frac{1}{2i\pi}\int^{\infty} e^{-\rho t}\left[e^{ix\sqrt{(\rho/\kappa)}} - e^{-ix\sqrt{(\rho/\kappa)}}\right]\frac{d\rho}{\rho},$$

i.e.
$$-\frac{1}{\pi}\int_0^{\infty} e^{-\rho t}\sin x\sqrt{(\rho/\kappa)}\,\frac{d\rho}{\rho},$$

i.e.
$$-\frac{2}{\pi}\int\limits_0^\infty e^{-\kappa u^2 t}\sin ux\,\frac{du}{u},$$

i.e.
$$-\frac{2}{\sqrt{\pi}}\int\limits^{x/2\sqrt{(\kappa t)}} e^{-u^2}\,du.\dagger$$

Also from the small circle we get 1.

Thus from (7) we have

$$v = v_0\left(1-\frac{2}{\sqrt{\pi}}\int\limits_0^{x/2\sqrt{(\kappa t)}} e^{-u^2}\,du\right)$$

$$= v_0\left(1-\operatorname{erf}\frac{x}{2\sqrt{(\kappa t)}}\right), \tag{8}$$

where $\operatorname{erf} x = \dfrac{2}{\sqrt{\pi}}\displaystyle\int\limits_0^x e^{-u^2}\,du.$

**40.** *Linear flow of heat in the solid $0 < x < l$: no flow across the boundary $x = 0$; the other boundary $x = l$ kept at a constant temperature $v_1$; the initial temperature of the solid $v_0$.*

Here we have to solve

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad 0 < x < l, \quad t > 0, \tag{1}$$

with
$$\frac{\partial v}{\partial x} = 0, \quad \text{when } x = 0,\ t > 0, \tag{2}$$

and
$$v = v_1, \quad \text{when } x = l,\ t > 0, \tag{3}$$

and
$$v = v_0, \quad \text{when } t = 0,\ 0 < x < l. \tag{4}$$

The subsidiary equation is

$$\frac{d^2\bar{v}}{dx^2}-\frac{p}{\kappa}\bar{v} = -\frac{v_0}{\kappa}, \quad 0 < x < l, \tag{5}$$

with
$$\frac{d\bar{v}}{dx} = 0, \quad \text{when } x = 0, \tag{6}$$

† This well-known integral can be obtained from

$$\int\limits_0^\infty e^{-a^2x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2a}e^{-b^2/a^2}$$

by integrating with respect to $b$ from 0 to $b$.

and $$\bar{v} = \frac{v_1}{p}, \quad \text{when } x = l. \tag{7}$$

Thus $$\bar{v} = \frac{v_0}{p} + \frac{v_1 - v_0}{p} \frac{\cosh x\sqrt{(p/\kappa)}}{\cosh l\sqrt{(p/\kappa)}}. \tag{8}$$

Then, by the Inversion Theorem,

$$v = v_0 + \frac{v_1 - v_0}{2i\pi} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}} \frac{d\lambda}{\lambda}. \tag{9}$$

The integrand is a single-valued function of $\lambda$ and there are poles at $\lambda = 0$, and $\lambda = -\kappa\left(\frac{2n-1}{2}\right)^2 \frac{\pi^2}{l^2}$, $n = 1, 2, \ldots$.

Consider the closed circuit of Fig. 10, where $AB$ is parallel to the imaginary axis at a distance $\gamma$ from it. If the circle has radius $R$ equal to $\kappa n^2\pi^2/l^2$, it will not pass through any pole of the integrand. As $n \to \infty$ the integral over the arc $BCA$ tends to zero; this will be proved in § 41.

Thus we can replace

$$\frac{1}{2i\pi} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}} \frac{d\lambda}{\lambda}$$

by the limit when $n \to \infty$ of this integral over the closed circuit $ABCA$. And by Cauchy's theorem this is equal to $2i\pi$ times the sum of the residues of the integrand at its poles.

The residue at the pole $\lambda = 0$ is 1.

The residue at the pole $\lambda = -\kappa(n - \frac{1}{2})^2\pi^2/l^2$ is

$$\frac{e^{-\kappa(n-\frac{1}{2})^2\pi^2 t/l^2} \cosh i(n - \frac{1}{2})\pi x/l}{\left[\lambda \frac{d}{d\lambda}\{\cosh l\sqrt{(\lambda/\kappa)}\}\right]_{\lambda = -\kappa(n-\frac{1}{2})^2\pi^2/l^2}}$$
$$= \frac{4(-1)^n}{\pi(2n-1)} e^{-\kappa(n-\frac{1}{2})^2\pi^2 t/l^2} \cos\frac{(2n-1)\pi x}{2l}.$$

Using these results in (9) we obtain

$$v = \overset{v_1}{\cancel{v_0}} + \frac{4(v_1 - v_0)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-\kappa(n-\frac{1}{2})^2\pi^2 t/l^2} \cos\frac{(2n-1)\pi x}{2l}.$$

**41.** *To show that when* $0 < x < l$ *and* $t > 0$,

$$\int e^{\lambda t}\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda}$$

*vanishes in the limit over the arc BCA of Fig.* 10.

We have to show that
$$\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\lambda\cosh l\sqrt{(\lambda/\kappa)}} \tag{1}$$
satisfies the conditions of the lemma of § 31 on the circle $\Gamma$ of radius $R = \kappa n^2\pi^2/l^2$.

Using the result
$$2\cosh(a+ib)\cosh(a-ib) = \cosh 2a+\cos 2b,$$
we find at the point $\lambda = Re^{i\theta}$
$$\begin{aligned}2|\cosh l\sqrt{(\lambda/\kappa)}|^2 &= \cosh(2n\pi\cos\tfrac{1}{2}\theta)+\cos(2n\pi\sin\tfrac{1}{2}\theta)\\ &= \cosh(2n\pi\cos\tfrac{1}{2}\theta)[1+\operatorname{sech}(2n\pi\cos\tfrac{1}{2}\theta)\cos(2n\pi\sin\tfrac{1}{2}\theta)].\end{aligned} \tag{2}$$

Let $\quad \sin\tfrac{1}{2}\beta = \dfrac{2n-\frac{1}{2}}{2n}\quad$ so that $\quad \cos\tfrac{1}{2}\beta = \dfrac{\sqrt{(8n-1)}}{4n}.$

When $\pi \geqslant \theta \geqslant \beta$,
$$2n\pi \geqslant 2n\pi\sin\tfrac{1}{2}\theta \geqslant 2n\pi\sin\tfrac{1}{2}\beta = (2n-\tfrac{1}{2})\pi,$$
and thus
$$\cos(2n\pi\sin\tfrac{1}{2}\theta) \geqslant 0.$$
Thus, when $\pi \geqslant \theta \geqslant \beta$,
$$2|\cosh l\sqrt{(\lambda/\kappa)}|^2 \geqslant \cosh(2n\pi\cos\tfrac{1}{2}\theta). \tag{3}$$
Also, when $\beta \geqslant \theta \geqslant 0$,
$$\begin{aligned}|\operatorname{sech}(2n\pi\cos\tfrac{1}{2}\theta)\cos(2n\pi\sin\tfrac{1}{2}\theta)| &\leqslant \operatorname{sech}(2n\pi\cos\tfrac{1}{2}\theta)\\ &\leqslant \operatorname{sech}(2n\pi\cos\tfrac{1}{2}\beta)\\ &= \operatorname{sech}\tfrac{1}{2}\pi\sqrt{(8n-1)}\\ &< \operatorname{sech}\tfrac{1}{2}\pi\sqrt{7}, \quad \text{when } n \geqslant 1\end{aligned} \tag{4}$$
Using (3) and (4) in (2), it follows that
$$2|\cosh l\sqrt{(\lambda/\kappa)}|^2 > (1-\operatorname{sech}\tfrac{1}{2}\pi\sqrt{7})\cosh(2n\pi\cos\tfrac{1}{2}\theta) \quad \text{when } \pi \geqslant \theta \geqslant 0.$$
Therefore
$$|\cosh l\sqrt{(\lambda/\kappa)}| > C\cosh^{\frac{1}{2}}(2n\pi\cos\tfrac{1}{2}\theta), \quad \pi \geqslant \theta \geqslant 0,$$
where $C$ is a constant, independent of $n$. Also
$$|\cosh x\sqrt{(\lambda/\kappa)}| = |\cosh\{(n\pi/l)x(\cos\tfrac{1}{2}\theta+i\sin\tfrac{1}{2}\theta)\}| \leqslant \cosh\{(n\pi/l)x\cos\tfrac{1}{2}\theta\}.$$
Hence
$$\begin{aligned}\left|\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\right| &< \frac{\cosh\{(n\pi/l)x\cos\frac{1}{2}\theta\}}{C\sqrt{[\cosh 2n\pi(\cos\frac{1}{2}\theta)]}}\\ &< C'\frac{e^{(n\pi/l)x\cos\frac{1}{2}\theta}}{e^{n\pi\cos\frac{1}{2}\theta}}\\ &< C'e^{-\{(l-x)/l\}n\pi\cos\frac{1}{2}\theta}\\ &< C', \quad \text{when } \pi \geqslant \theta \geqslant 0.\end{aligned}$$
And this holds also for $\pi \leqslant \theta \leqslant 2\pi$.

Thus the conditions of the lemma of § 31 are satisfied and the result follows.

**42.** *A stretched string with its ends fixed at the origin and $x = l$ is plucked at its middle point and released at $t = 0$. If the displacement is $b$, find the subsequent vibration.*

Here we have to solve

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \quad 0 < x < l, \quad t > 0, \tag{1}$$

with

$$\left.\begin{aligned} y &= \frac{2b}{l}x, \quad 0 < x < \tfrac{1}{2}l, \\ y &= \frac{2b}{l}(l-x), \quad \tfrac{1}{2}l < x < l, \end{aligned}\right\} \quad t = 0, \tag{2}$$

$$\frac{\partial y}{\partial t} = 0, \quad 0 < x < l, \quad t = 0, \tag{3}$$

$$y \quad 0, \quad \text{when } x = 0 \text{ and } x = l, \quad t > 0. \tag{4}$$

Multiplying (1) by $e^{-pt}$ $(p > 0)$, integrating with regard to $t$ from 0 to $\infty$, and using (2) and (3), we obtain the subsidiary equation

$$c^2\frac{d^2\bar{y}}{dx^2} - p^2\bar{y} = -pf(x), \quad 0 < x < l, \tag{5}$$

where

$$\begin{aligned} f(x) &= \frac{2b}{l}x, \quad 0 \leqslant x \leqslant \tfrac{1}{2}l, \\ &= \frac{2b}{l}(l-x), \quad \tfrac{1}{2}l \leqslant x \leqslant l, \end{aligned}$$

and

$$\bar{y} = 0, \quad \text{when } x = 0 \text{ and } x = l. \tag{6}$$

We solve (5) and (6) by the method of variation of parameters,† as follows:

Let
$$\bar{y} = A\cosh qx + B\sinh qx, \tag{7}$$

where $q = p/c$ and $A$, $B$ are functions of $x$ to be determined.

Then
$$\bar{y}' = q(A\sinh qx + B\cosh qx)$$

provided that
$$A'\cosh qx + B'\sinh qx = 0.$$

Also

$$\bar{y}'' = q^2(A\cosh qx + B\sinh qx) + q(A'\sinh qx + B'\cosh qx).$$

† For an alternative method, using the ideas of Chapter I, see Appendix IV, Ex. 3.

Thus $$\bar{y}''-q^2\bar{y} = q(A'\sinh qx+B'\cosh qx)$$

and so $\bar{y} = A\cosh qx+B\sinh qx$ satisfies (5)

if
$$A'\cosh qx+B'\sinh qx = 0 \tag{8}$$

and
$$A'\sinh qx+B'\cosh qx = -\frac{f(x)}{c}. \tag{9}$$

From (8) and (9)
$$\left.\begin{aligned} A' &= \frac{f(x)}{c}\sinh qx \\ \text{and}\qquad B' &= -\frac{f(x)}{c}\cosh qx. \end{aligned}\right\} \tag{10}$$

But from (6)
$$A(0) = 0.$$

Therefore
$$A(x) = \frac{1}{c}\int\limits_0^x f(\xi)\sinh q\xi\, d\xi. \tag{11}$$

Also
$$A(l)\cosh ql+B(l)\sinh ql = 0.$$

Therefore
$$B(l) = -\frac{1}{c}\coth ql\int\limits_0 f(\xi)\sinh q\xi\, d\xi. \tag{12}$$

Thus, from (10) and (12),
$$B(x) = \frac{1}{c}\left[\int\limits_x^l f(\xi)\cosh q\xi\, d\xi -\coth ql\int\limits_0^l f(\xi)\sinh q\xi\, d\xi\right]. \tag{13}$$

Finally, from (7), (11), and (13),
$$c\sinh ql\bar{y} = \sinh q(l-x)\int\limits_0^x f(\xi)\sinh q\xi\, d\xi +$$
$$+\sinh qx\int^l f(\xi)\sinh q(l-\xi)\, d\xi. \tag{14}$$

Then, substituting for $f(x)$ from (2), we have
$$\frac{l}{2b}\bar{y} = \frac{x}{p}-\frac{c}{p^2}\frac{\sinh qx}{\cosh\frac{1}{2}ql},\quad 0 \leqslant x \leqslant \tfrac{1}{2}l, \tag{15}$$

and
$$\frac{l}{2b}\bar{y} = \frac{l-x}{p}-\frac{c}{p^2}\frac{\sinh q(l-x)}{\cosh\frac{1}{2}ql},\quad \tfrac{1}{2}l \leqslant x \leqslant l. \tag{16}$$

We have now to find $y$, and from symmetry it is clear that we need only deal with (15).

The first term on the right gives $x$.

For $\frac{1}{p^2}\frac{\sinh qx}{\cosh \frac{1}{2}ql}$ we use the Inversion Theorem and have

$$\frac{1}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\sinh(\lambda x/c)}{\cosh(\lambda l/2c)}\frac{d\lambda}{\lambda^2}. \tag{17}$$

We now consider this integral taken over the closed circuit of Fig. 10. The circle $\Gamma$ of radius $R$ equal to $2n\pi c/l$ does not pass through any pole of the integrand as these are at $\lambda = 0$ and $\lambda = \pm i\frac{(2n-1)\pi c}{l}$, $n = 1, 2,\ldots$.

As $n \to \infty$ the integral over the arc $BCA$ tends to zero; this will be proved in § 43. We can thus replace (17) by the limit when $n \to \infty$ of this integral over the closed circuit $ABCA$.

Then from the Theory of Residues we obtain the value of (17) as an infinite series.

$$\text{The pole at } \lambda = 0 \text{ gives } x/c. \tag{18}$$

The pole at $\lambda = i\frac{(2n-1)\pi c}{l}$ gives

$$e^{i(2n-1)\pi ct/l}\frac{\sinh i(2n-1)\pi x/l}{\frac{d}{d\lambda}\left[\lambda^2\cosh\frac{\lambda l}{2c}\right]_{\lambda=i(2n-1)\pi c/l}}$$

i.e.

$$(-1)^n\frac{2l}{\pi^2 c}\frac{\sin(2n-1)\pi x/l}{(2n-1)^2}e^{i(2n-1)\pi ct/l}. \tag{19}$$

So the poles at $\pm i(2n-1)\pi c/l$ give

$$(-1)^n\frac{4l}{\pi^2 c}\frac{\sin(2n-1)\frac{\pi x}{l}\cos(2n-1)\frac{\pi ct}{l}}{(2n-1)^2} \tag{20}$$

Then from (15) we have

$$\frac{ly}{2b} = x-\left\{x+\frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l}\cdot\cos\frac{(2n-1)\pi ct}{l}\right\}.$$

Thus†

$$y = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}, \tag{21}$$

when $0 \leqslant x \leqslant \frac{1}{2}l$.

For $\frac{1}{2}l \leqslant x \leqslant l$, we replace $x$ in (21) by $(l-x)$.

**43.** *To show that when* $0 < x < \frac{1}{2}l$ *and* $t > 0$

$$\int e^{\lambda t}\frac{\sinh(\lambda x/c)}{\cosh(\lambda l/2c)}\frac{d\lambda}{\lambda^2}$$

*vanishes in the limit over the arc BCA of Fig.* 10.

If we can show that $$\frac{\sinh(\lambda x/c)}{\cosh(\lambda l/2c)}$$

is bounded on the circle $\Gamma$ of radius $R = 2n\pi c/l$, the result will follow from the lemma of § 31. We prove this for $\lambda$ in the first quadrant, $\frac{1}{2}\pi \geqslant \theta \geqslant 0$; the other quadrants are treated similarly.

If $\lambda = Re^{i\theta}$, where $R = 2n\pi c/l$, we have

$$\cosh\frac{\lambda l}{2c} = \cosh(2n\pi\cos\theta)[1+\operatorname{sech}(2n\pi\cos\theta)\cos(2n\pi\sin\theta)]. \tag{1}$$

Now let $$\cos\beta = \frac{2n-\frac{1}{2}}{2n}, \quad 0 < \beta < \tfrac{1}{2}\pi,$$

so that $$\sin\beta = \frac{\sqrt{(8n-1)}}{4n}.$$

Then, if $\frac{1}{2}\pi \geqslant \theta \geqslant \frac{1}{2}\pi-\beta$,

$$2n\pi \geqslant 2n\pi\sin\theta \geqslant 2n\pi\cos\beta = (2n-\tfrac{1}{2})\pi,$$

and thus $$\cos(2n\pi\sin\theta) \geqslant 0.$$

Therefore $$2|\cosh(\lambda l/2c)|^2 \geqslant \cosh(2n\pi\cos\theta). \tag{2}$$

Also, if $\frac{1}{2}\pi-\beta \geqslant \theta \geqslant 0$,

$$\begin{aligned}|\operatorname{sech}(2n\pi\cos\theta)\cos(2n\pi\sin\theta)| &\leqslant \operatorname{sech}(2n\pi\cos\theta)\\ &\leqslant \operatorname{sech}(2n\pi\sin\beta)\\ &= \operatorname{sech}\tfrac{1}{2}\pi\sqrt{(8n-1)}\\ &< \operatorname{sech}\tfrac{1}{2}\pi\sqrt{7}, \quad \text{when } n > 1.\end{aligned} \tag{3}$$

† It should be remarked that for this problem it is not possible to verify directly that the solution obtained as a line integral satisfies the differential equation and initial and boundary conditions. It will be found that the solutions obtained from (15) and (16) by the Inversion Theorem cannot be twice differentiated under the integral sign and thus we cannot verify by direct differentiation that they satisfy the differential equation. The difficulty is due to the fact that the given initial form of the string has a discontinuous derivative at $x = \frac{1}{2}l$. The solution obtained by the Wave Method in § 44 can be verified.

Using (2) and (3) in (1) we obtain

$$2|\cosh(\lambda l/2c)|^2 > (1-\operatorname{sech}\tfrac{1}{2}\pi\sqrt{7})\cosh(2n\pi\cos\theta), \quad \tfrac{1}{2}\pi \geqslant \theta \geqslant 0,$$

and thus

$$|\cosh(\lambda l/2c)| > C\cosh^{\frac{1}{2}}(2n\pi\cos\theta), \quad \tfrac{1}{2}\pi \geqslant \theta \geqslant 0.$$

But
$$\sinh \lambda x \leqslant \cosh\left(\frac{2n\pi x}{l}\cos\theta\right).$$

Hence
$$\frac{\sinh(\lambda x/c)}{\cosh(\lambda l/2c)} \quad \frac{\cosh\{(2n\pi x/l)\cos\theta\}}{C\cosh^{\frac{1}{2}}(2n\pi\cos\theta)}$$
$$< C'e^{-(2n\pi/l)(\frac{1}{2}l-x)\cos\theta} < C',$$

where $C'$ is a constant. This is the result required.

**44.** The solution obtained in § 42 (21) can be put in another form which shows the shape the string assumes as $t$ changes.† Since $y$ is periodic in $t$ of period $2l/c$, we need only consider the interval $0 < ct \leqslant 2l$.

First we prove the following lemma:

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{a\lambda}\frac{d\lambda}{\lambda^2} = 2i\pi a, \quad \textit{when } a > 0,$$
$$= 0, \qquad \textit{when } a \leqslant 0.$$

(i) When $a > 0$, it follows from the lemma of § 31 that

$$\int e^{a\lambda}\frac{d\lambda}{\lambda^2}$$

taken over the arc $BCA$ of Fig. 10 tends to zero when the radius $R$ of this circle tends to infinity.

Also there is a pole at $\lambda = 0$. Thus

$$\frac{1}{2i\pi}\int e^{a\lambda}\frac{d\lambda}{\lambda^2} = a,$$

when the integral is taken over $ABCA$ and the radius $R \to \infty$.

Therefore
$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{a\lambda}\frac{d\lambda}{\lambda^2} = 2i\pi a.$$

† Heaviside made frequent use of this device, which he called the Wave Method. Cf. *E.M.T.* **2**, 71.

(ii) When $a \leqslant 0$, we take the closed circuit $ABHA$ of Fig. 17, p. 200. There is no pole inside this circuit, and an argument similar to that of § 31 shows that when the radius tends to infinity the integral over $BHA$ tends to zero.

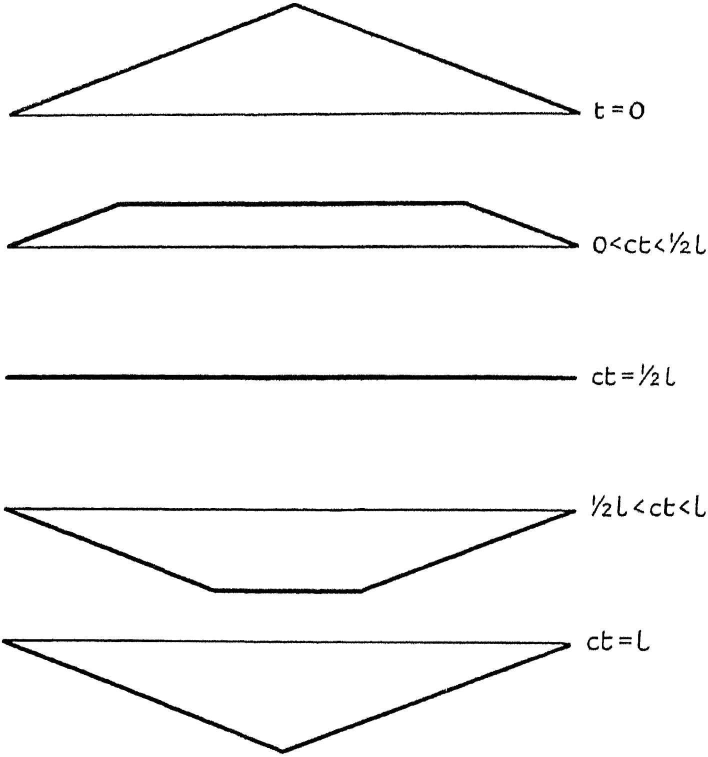


FIG. 12

It follows that
$$\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{a\lambda}\frac{d\lambda}{\lambda^2} = 0.$$

We now return to § 42 (15). This gave, by the Inversion Theorem,
$$\frac{ly}{2b} = x - \frac{c}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\sinh(\lambda x/c)}{\cosh(\lambda l/2c)}\frac{d\lambda}{\lambda^2}. \tag{1}$$

But

$$\frac{\sinh(\lambda x/c)}{\cosh(\lambda l/2c)} = e^{-\lambda(\frac{1}{2}l-x)/c}\frac{1-e^{-2\lambda x/c}}{1+e^{-\lambda l/c}}$$

$$= [e^{-\lambda(\frac{1}{2}l-x)/c}-e^{-\lambda(\frac{1}{2}l+x)/c}]\sum_{m=0}^{\infty}(-1)^m e^{-m\lambda l/c}.$$

Thus

$$\frac{1}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\sinh(\lambda x/c)}{\cosh(\lambda l/2c)}\frac{d\lambda}{\lambda^2}$$

$$= \frac{1}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} [e^{\lambda(ct-\frac{1}{2}l+x)/c}-e^{\lambda(ct-\frac{1}{2}l-x)/c}]\sum_{m=0}^{\infty}(-1)^m e^{-m\lambda l/c}\frac{d\lambda}{\lambda^2}. \quad (2)$$

We can integrate this series term by term, since it is uniformly convergent, and obtain

$$\frac{1}{2i\pi}\sum_{m=0}^{\infty}(-1)^m\int\limits_{\gamma-i\infty}^{\gamma+i\infty}[e^{\lambda\{ct-(m+\frac{1}{2})l+x\}/c}-e^{\lambda\{ct-(m+\frac{1}{2})l-x\}/c}]\frac{d\lambda}{\lambda^2}.$$

The result is obtained by applying the lemma above to the successive terms of this series.

It will be convenient to take $t$

(I) in the interval $0 < ct \leqslant \frac{1}{2}l$,

(II) in the interval $\frac{1}{2}l < ct \leqslant l$.

I. The term $e^{\lambda\{x-(\frac{1}{2}l-ct)\}/c}$ in (2) gives zero when $x \leqslant \frac{1}{2}l-ct$ by the above lemma.

It gives $(ct-\frac{1}{2}l+x)/c$ when $x > \frac{1}{2}l-ct$.

The term $e^{-\lambda\{x+(\frac{1}{2}l-ct)\}/c}$ gives zero.

And all the other terms give zero.

Thus, when $0 < ct \leqslant \frac{1}{2}l$,

$$\frac{l}{2b}y = x, \quad \text{when } x \leqslant \tfrac{1}{2}l-ct,$$

$$= x-(ct-\tfrac{1}{2}l+x), \quad \text{when } x > \tfrac{1}{2}l-ct,$$

$$= \tfrac{1}{2}l-ct,$$

i.e.

$$y = \frac{2b}{l}x, \quad \text{when } x \leqslant \tfrac{1}{2}l-ct,$$

$$y = \frac{2b}{l}(\tfrac{1}{2}l-ct), \quad \text{when } x > \tfrac{1}{2}l-ct.$$

When $ct = \frac{1}{2}l$, $y = 0$.

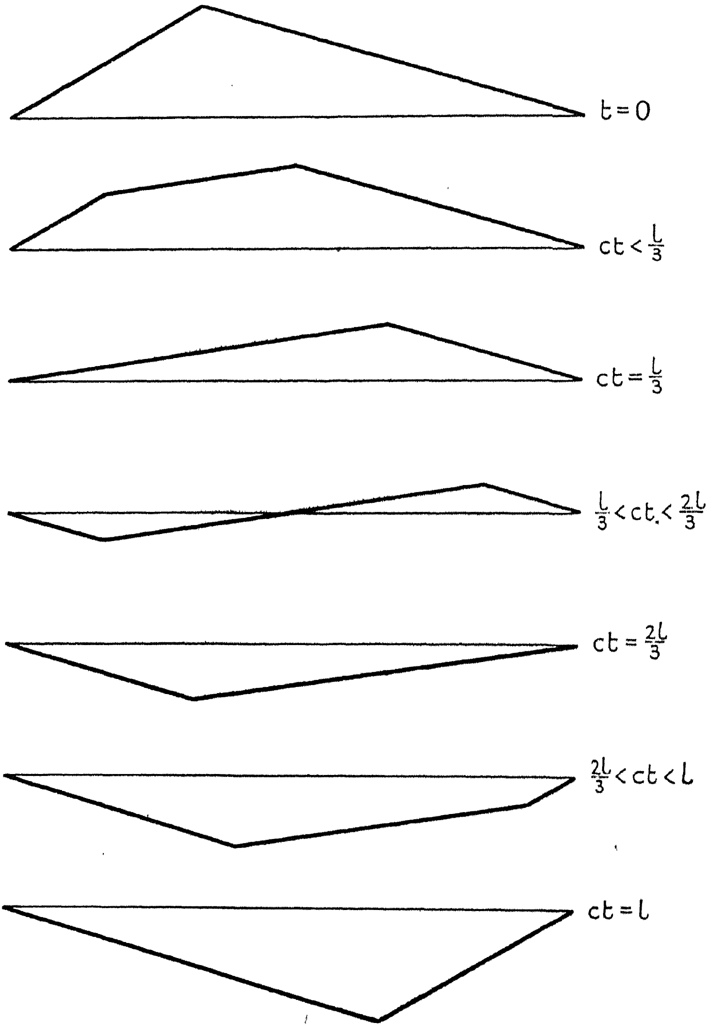


FIG. 13

II. The term $e^{\lambda(ct-\frac{1}{2}l+x)/c}$ gives $(ct-\frac{1}{2}l+x)/c$.

The term $e^{\lambda(ct-\frac{1}{2}l-x)/c}$ gives $(ct-\frac{1}{2}l-x)/c$ when $x < ct-\frac{1}{2}l$, and it gives zero when $x \geqslant ct-\frac{1}{2}l$.

All the other terms give zero.

Thus, when $\frac{1}{2}l < ct \leqslant l$,

$$\frac{l}{2b}y = x - \{(ct - \tfrac{1}{2}l + x) - (ct - \tfrac{1}{2}l - x)\}, \quad \text{when } x < ct - \tfrac{1}{2}l,$$

$$= x - (ct - \tfrac{1}{2}l + x), \quad \text{when } x \geqslant ct - \tfrac{1}{2}l,$$

i.e.

$$y = -\frac{2b}{l}x, \quad \text{when } x < ct - \tfrac{1}{2}l,$$

$$= \frac{2b}{l}(\tfrac{1}{2}l - ct), \quad \text{when } x \geqslant ct - \tfrac{1}{2}l.$$

When $ct = l$, $y = -\dfrac{2b}{l}x$.

As $ct$ passes from $l$ to $2l$, this motion takes place in the reversed order.

It will be seen that, except when $ct$ is a multiple of $\frac{1}{2}l$, the form of the string consists of three straight portions: the outer having the same gradients as the two pieces into which the string was initially displaced, while the middle portion is parallel to the axis of $x$. The middle portion moves parallel to itself with velocity $2bc/l$ and its ends have a velocity $c$ parallel to the axis of $x$.

It is clear that except at the corners $\partial^2 y/\partial x^2$ is zero and $\partial^2 y/\partial t^2$ is also zero. So the equation of the problem is satisfied except at these points, and the initial and boundary conditions are also satisfied.

The result for the general case in which the string is plucked at the point $x = \alpha$ will follow in the same way for any $\alpha < l$. It is simpler to take $\alpha < \frac{1}{2}l$ and in the discussion the intervals

$$0 < ct \leqslant \alpha, \qquad \alpha < ct \leqslant l - \alpha, \qquad l - \alpha < ct < l$$

would be considered in place of the equal intervals for $\alpha = \frac{1}{2}l$.

The form of the string plucked at its mid-point is shown in Fig. 12, and that of the string plucked at a point of trisection in Fig. 13.

## CHAPTER VI

# CONDUCTION OF HEAT

**45.** The rate of flow of heat in a homogeneous solid across a surface is $-K\dfrac{\partial v}{\partial n}$ per unit area, where $v$ is the temperature and $K$ a constant called the thermal conductivity, $\partial/\partial n$ denoting differentiation along the normal. Taking as an element of the solid at the point $P(x, y, z)$ a rectangular parallelepiped with $P$ as centre and edges parallel to the coordinate axes, of lengths $dx$, $dy$, and $dz$, we find that the rate of flow of heat into the element is

$$K\nabla^2 v\, dxdydz. \tag{1}$$

But the element is gaining heat at the rate

$$\rho c \frac{\partial v}{\partial t} dxdydz, \tag{2}$$

where $\rho$ is the density and $c$ the specific heat. Thus, if there is no gain of heat in the element other than by conduction, we have

$$\frac{\partial v}{\partial t} = \kappa \nabla^2 v, \tag{3}$$

where $\kappa = K/c\rho$.

If heat is being produced at $(x, y, z)$ in any other way, a term must be added to the right-hand side of (3).

In this chapter we shall discuss various problems in the conduction of heat where the flow is either linear and equation (3) reduces to

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}, \tag{4}$$

or the distribution of temperature has cylindrical or spherical symmetry, so that equation (3) reduces to

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) \tag{5}$$

or

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}\right). \tag{6}$$

There will also be initial and boundary conditions which the temperature must satisfy.

**46.** Before proceeding farther it is advisable to mention some fundamental solutions of the equation of conduction which we shall have occasion to use in some of the following sections.†

First we note that § 45 (3) is satisfied by

$$v = \frac{Q}{\{2\sqrt{(\pi\kappa t)}\}^3} e^{-\{(x-x')^2+(y-y')^2+(z-z')^2\}/4\kappa t}, \tag{1}$$

and it will be seen that, when $t \to 0$, this value of $v \to 0$ at all points except $(x', y', z')$, where it becomes infinite. But

$$\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} v \, dxdydz = Q, \quad \text{when } t > 0,$$

since each of the integrals

$$\frac{1}{2\sqrt{(\pi\kappa t)}} \int\limits_{-\infty}^{\infty} e^{-(x-x')^2/4\kappa t} \, dx', \qquad \frac{1}{2\sqrt{(\pi\kappa t)}} \int\limits_{-\infty}^{\infty} e^{-(y-y')^2/4\kappa t} \, dy',$$

$$\frac{1}{2\sqrt{(\pi\kappa t)}} \int\limits_{-\infty}^{\infty} e^{-(z-z')^2/4\kappa t} \, dz'$$

is unity.

The solution in (1) is said to be *the temperature due to an instantaneous point source of strength $Q$ at $(x', y', z')$ at $t = 0$*, since the quantity of heat instantaneously generated at the point is $Q\rho c$.

Similarly,
$$v = \frac{Q}{4\pi\kappa t} e^{-\{(x-x')^2+(y-y')^2\}/4\kappa t} \tag{2}$$

satisfies
$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

and it tends to zero when $t \to 0$ at all points except $(x', y')$, where it becomes infinite.

Also, when $t > 0$,

$$\int\limits^{\infty} \int\limits^{\infty} v \, dxdy = Q.$$

† Cf. Carslaw, *Conduction of Heat* (2nd ed., 1921), chap. ix. This work will be referred to in future as *C.H.*

Again,
$$v = \frac{Q}{2\sqrt{(\pi\kappa t)}} e^{-(x-x')^2/4\kappa t} \tag{3}$$
satisfies
$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2},$$
and it tends to zero when $t \to 0$ at all points except $x'$, where it becomes infinite.

Also
$$\int\limits_{-\infty}^{\infty} v\,dx = Q, \quad \text{when } t > 0.$$

Equation (2) can be obtained from the point source in three dimensions by integrating
$$\frac{V\,dx'dy'dz'}{[2\sqrt{(\pi\kappa t)}]^3} e^{-\{(x-x')^2+(y-y')^2+(z-z')^2\}/4\kappa t} \tag{4}$$
with regard to $z'$ from $-\infty$ to $\infty$, and finally substituting $Q$ for $V\,dx'dy'$. From this point of view (2) can be said to be *the temperature due to an instantaneous line source through* $(x', y')$ *at* $t = 0$ *of strength* $Q$ *per unit length.*

Similarly, (3) can be obtained from (4) by integrating with regard to $y'$ and $z'$ from $-\infty$ to $\infty$ and finally substituting $Q$ for $V\,dx'$; and, from this point of view, (3) can be called *the temperature due to an instantaneous plane source through* $x'$ *at* $t = 0$ *of strength* $Q$ *per unit area.*

Other important solutions of the equation of conduction are those for the *Instantaneous Cylindrical Surface Source* and *Spherical Surface Source.*

In the former we are dealing with flow in two dimensions, and we obtain our solution by integrating
$$\frac{Vr'\,dr'd\theta'}{4\pi\kappa t} e^{-(r^2+r'^2-2rr'\cos\theta')/4\kappa t}$$
with regard to $\theta'$ from 0 to $2\pi$. This gives
$$v = \frac{Vr'\,dr'}{4\pi\kappa t} e^{-(r^2+r'^2)/4\kappa t} \int\limits^{2\pi} e^{(rr'/2\kappa t)\cos\theta'}\,d\theta$$
$$\frac{Vr'\,dr'}{2\kappa t} e^{-(r^2+r'^2)/4\kappa t} I_0\left(\frac{rr'}{2\kappa t}\right),$$

and, writing $Q = 2\pi r' V\,dr'$,

$$v = \frac{Q}{4\pi\kappa t} e^{-(r^2+r'^2)/4\kappa t} I_0\left(\frac{rr'}{2\kappa t}\right). \tag{5}$$

It will be seen that (5) satisfies

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right),$$

the form the equation of conduction takes for cylindrical symmetry. Also $v$ tends to zero when $t \to 0$ at all points except on the cylinder $r = r'$, where it becomes infinite.

But, using Weber's First Integral,†

$$\int_0^\infty e^{-a^2u^2} J_0(bu)u\,du = \frac{1}{2a^2} e^{-b^2/4a^2}$$

we have
$$2\pi \int_0^\infty vr\,dr = Q, \quad \text{when } t > 0.$$

We can thus regard (5) as the temperature due to a quantity of heat $Q\rho c$ per unit length instantaneously generated on the surface of the cylinder $r = r'$ at $t = 0$.

Again, with spherical polar coordinates, integrating

$$v = \frac{Vr'^2\,dr'\sin\theta'\,d\theta' d\phi'}{[2\sqrt{(\pi\kappa t)}]^3} e^{-(r^2+r'^2-2rr'\cos\theta')/4\kappa t}$$

over the sphere of radius $r'$, we have

$$\begin{aligned} v &= \frac{Vr'^2\,dr'}{4\pi^{\frac{1}{2}}(\kappa t)^{\frac{3}{2}}} e^{-(r^2+r'^2)/4\kappa t} \int_0^\pi e^{(rr'/2\kappa t)\cos\theta'} \sin\theta'\,d\theta' \\ &= \frac{Vr'^2\,dr'}{4\pi^{\frac{1}{2}}(\kappa t)^{\frac{3}{2}}} e^{-(r^2+r'^2)/4\kappa t} \int_{-1}^1 e^{(rr'/2\kappa t)\mu}\,d\mu \\ &= \frac{Vr'\,dr'}{2r(\pi\kappa t)^{\frac{1}{2}}} \{e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t}\}, \end{aligned}$$

and, writing $Q = 4\pi r'^2 V\,dr'$,

$$v = \frac{Q}{8\pi rr'(\pi\kappa t)^{\frac{1}{2}}} [e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t}]. \tag{6}$$

† Watson, *Theory of Bessel Functions* (1922), § 13.3 (1); Gray and Mathews, *Treatise on Bessel Functions* (2nd ed., 1922), p. 68 (15). These works will be referred to in future as *W.B.F.*, and *G. and M.*

It will be seen that (6) satisfies

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}\right),$$

the form the equation of conduction takes for spherical symmetry. Also $v$ tends to zero when $t \to 0$ at all points except on the sphere $r = r'$, where it becomes infinite.

But $$4\pi\int\limits_0^\infty vr^2\,dr = Q, \quad \text{when } t > 0.$$

Thus we can regard (6) as the temperature due to a quantity of heat $Q\rho c$ instantaneously generated on the surface of the sphere $r = r'$ at $t = 0$.

The solutions obtained in (5) and (6) are said to be those for *an Instantaneous Cylindrical Surface Source of Strength Q* and *an Instantaneous Spherical Surface Source of Strength Q* respectively.

## Linear Flow

**47.** *Flow of heat in a semi-infinite solid, $x > 0$. The boundary $x = 0$ kept at $a\cos\omega t$, $t > 0$. The initial temperature of the solid zero.*

The equations for the temperature $v$ at $x$ at the time $t$ are

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x > 0,\ t > 0, \tag{1}$$

$$v = 0, \quad \text{when } t = 0,\ x > 0, \tag{2}$$

$$v = a\cos\omega t, \quad \text{when } x = 0,\ t > 0. \tag{3}$$

Then the subsidiary equation is

$$\kappa\frac{d^2\bar{v}}{dx^2}-p\bar{v} = 0, \quad x > 0, \tag{4}$$

with $$\bar{v} = \frac{ap}{p^2+\omega^2}, \quad \text{when } x = 0. \tag{5}$$

From (4) and (5)

$$\bar{v} = \frac{ap}{p^2+\omega^2}e^{-\sqrt{(p/\kappa)}x}, \tag{6}$$

and we have to find $v$ from $\bar{v} = \int\limits_0^\infty e^{-pt}v\,dt$, i.e.

$$\frac{ap}{p^2+\omega^2}e^{-\sqrt{(p/\kappa)}x} = \int\limits^\infty e^{-pt}v\,dt. \tag{7}$$

Using the Inversion Theorem,

$$v = \frac{a}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t-\sqrt{(\lambda/\kappa)}x}\frac{\lambda}{\lambda^2+\omega^2}\,d\lambda. \tag{8}$$

There is a branch-point at the origin, so we use the contour of Fig. 11, § 39. The integrand is single-valued in the region bounded by this closed circuit and has poles at $\pm i\omega$.

It is easy to show† that when $R \to \infty$ the integrals over $BB'$ and $B'F$ tend to zero.

The same remark applies to the integrals over $CA'$ and $A'A$.

† *To evaluate* $\lim\limits_{R\to\infty} I$, *when* $I = \int e^{\lambda t-\sqrt{(\lambda/\kappa)}x}\frac{\lambda\,d\lambda}{\lambda^2+\omega^2}$, *taken over the arcs* $BB'$ *and* $B'F$ *of the circle* $\Gamma$ *of radius* $R$.

[See FIG. 14, p. 112.]

Let
$$\beta = \cos^{-1}\frac{\gamma}{R},$$
$$R_1 = |Re^{i\theta}-i\omega|, \quad \pi \geqslant \theta \geqslant \beta,$$
$$R_2 = |Re^{i\theta}+i\omega|, \quad \pi \geqslant \theta \geqslant \beta.$$

Then
$$\frac{R}{R_2} < 1.$$

Also
$$R_1 \geqslant R-\omega;$$

thus
$$\frac{R}{R_1} \leqslant \frac{R}{R-\omega} < 2, \quad \text{if } R > 2\omega.$$

Therefore
$$\frac{R^2}{R_1R_2} < 2, \quad \text{if } R > 2\omega.$$

*Over* $BB'$.
$$|I| < e^{\gamma t}\frac{R^2}{R_1R_2}\int\limits_\beta^{\frac{1}{2}\pi} d\theta$$
$$< 2e^{\gamma t}(\tfrac{1}{2}\pi-\beta) = 2e^{\gamma t}\sin^{-1}\frac{\gamma}{R}.$$

Thus $\lim\limits_{R\to\infty} I = 0.$

*Over* $B'F$.
$$|I| < \frac{R^2}{R_1R_2}\int\limits_{\frac{1}{2}\pi}^{\pi} e^{Rt\cos\theta}\,d\theta$$
$$< 2\int\limits_0^{\frac{1}{2}\pi} e^{-Rt\sin\theta}\,d\theta < 2\int\limits^{\frac{1}{2}\pi} e^{-2Rt\theta/\pi}\,d\theta < \frac{\pi}{Rt}.$$

Thus $\lim\limits_{R\to\infty} I = 0.$

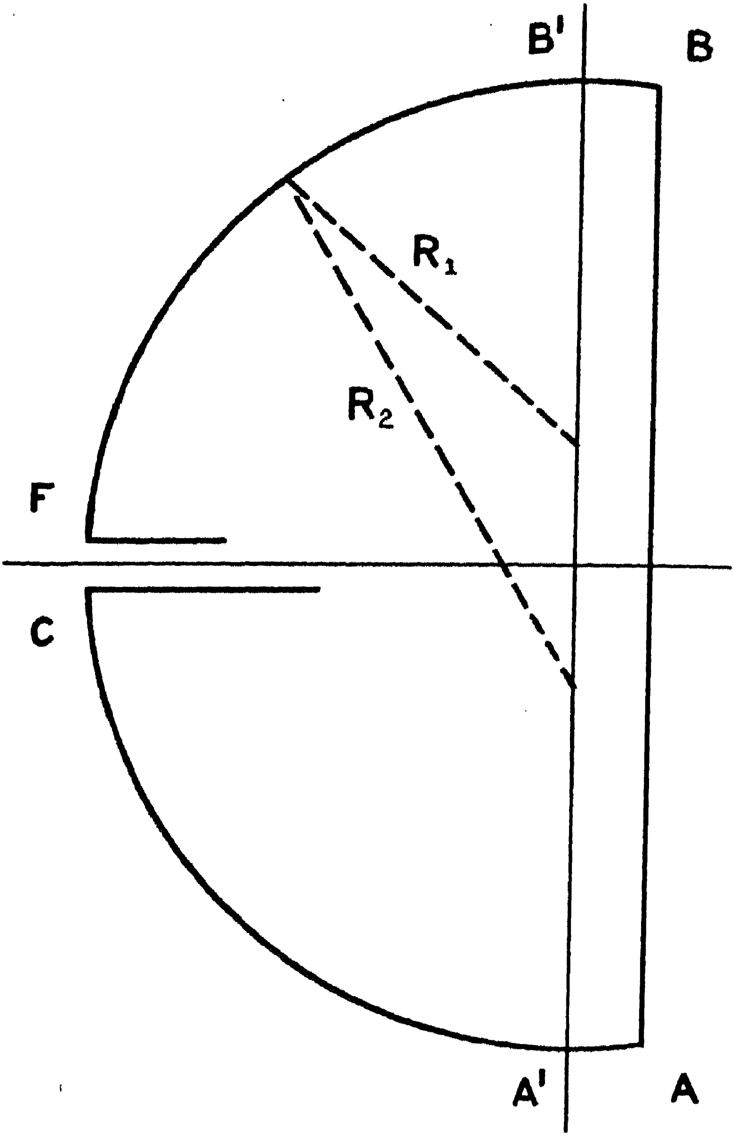


FIG. 14

Then, by Cauchy's theorem,

$$\frac{a}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t-\sqrt{(\lambda/\kappa)}x}\frac{\lambda\,d\lambda}{\lambda^2+\omega^2}$$

$$=\frac{a}{2i\pi}\begin{pmatrix}\text{sum of the integrals over } CD, \text{ the small circle,}\\ \text{and } EF \text{ when } R\to\infty \text{ and } \epsilon\to 0\end{pmatrix}+$$

$$+a\frac{e^{i\omega t}}{2}e^{-\sqrt{(\omega/\kappa)}x(\cos\frac{1}{4}\pi+i\sin\frac{1}{4}\pi)}+a\frac{e^{-i\omega t}}{2}e^{-\sqrt{(\omega/\kappa)}x(\cos\frac{1}{4}\pi-i\sin\frac{1}{4}\pi)}.$$

To evaluate the integral over $CD$, put $\lambda = \rho e^{-i\pi}$ and we obtain, when $R\to\infty$ and $\epsilon\to 0$,

$$-\frac{a}{2i\pi}\int\limits_0^\infty e^{-\rho t}e^{i\sqrt{(\rho/\kappa)}x}\frac{\rho}{\rho^2+\omega^2}\,d\rho.$$

Similarly, with $\lambda = \rho e^{i\pi}$ for $EF$, we obtain

$$\frac{a}{2i\pi}\int\limits_0^\infty e^{-\rho t}e^{-i\sqrt{(\rho/\kappa)}x}\frac{\rho}{\rho^2+\omega^2}\,d\rho.$$

And the small circle gives zero when $\epsilon\to 0$.

It follows that

$$v = ae^{-\sqrt{(\omega/2\kappa)}x}\cos\{\omega t-\sqrt{(\omega/2\kappa)}x\}-\frac{a}{\pi}\int\limits_0^\infty e^{-\rho t}\sin\sqrt{(\rho/\kappa)}x\,\frac{\rho\,d\rho}{\rho^2+\omega^2}. \quad (9)$$

**48.** *The same solid: radiation at $x = 0$ into a medium at zero; initial temperature of the solid $v_0$.*

Here the equations for $v$ are

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x>0,\ t>0, \quad (1)$$

$$v = v_0, \quad \text{when } t=0,\ x>0, \quad (2)$$

$$-\frac{\partial v}{\partial x}+hv = 0, \quad \text{when } x=0,\ t>0. \quad (3)$$

Then, as in the preceding section, the subsidiary equation is

$$\kappa\frac{d^2\bar v}{dx^2}-p\bar v = -v_0, \quad x>0, \quad (4)$$

with
$$\frac{d\bar{v}}{dx}+h\bar{v}=0, \quad \text{when } x=0, \tag{5}$$
where $\bar{v}=\int\limits_0^\infty e^{-pt}v\,dt$.

Solving (4) and (5),
$$\bar{v}=\frac{v_0}{p}\left(1-\frac{h}{\sqrt{(p/\kappa)}+h}e^{-\sqrt{(p/\kappa)}x}\right),$$
i.e.
$$\bar{v}=\frac{v_0}{p}(1-e^{-\sqrt{(p/\kappa)}x})+\frac{v_0}{\sqrt{(p\kappa)}\{\sqrt{(p/\kappa)}+h\}}e^{-\sqrt{(p/\kappa)}x}. \tag{6}$$
We know† that
$$\frac{1}{p}(1-e^{-\sqrt{(p/\kappa)}x})=\int\limits_0^\infty e^{-pt}\left[\frac{2}{\sqrt{\pi}}\int\limits_0^{x/2\sqrt{(\kappa t)}}e^{-u^2}\,du\right]dt. \tag{7}$$
We have now to find $F(t)$, where
$$\frac{1}{\sqrt{p}\{\sqrt{(p/\kappa)}+h\}}e^{-\sqrt{(p/\kappa)}x}=\int\limits_0^\infty e^{-pt}F(t)\,dt.$$
Using the Inversion Theorem,
$$F(t)=\frac{1}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t-\sqrt{(\lambda/\kappa)}x}\frac{d\lambda}{\sqrt{\lambda}\{\sqrt{(\lambda/\kappa)}+h\}}.$$

We take again the closed circuit of Fig. 11 and note that the integral round the closed circuit is zero, and that when $R\to\infty$, the integrals over the arcs of $\Gamma\to 0$. Thus
$$\frac{1}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t-\sqrt{(\lambda/\kappa)}x}\frac{d\lambda}{\sqrt{\lambda}\{\sqrt{(\lambda/\kappa)}+h\}}$$
$$=\frac{1}{2i\pi}\left(\begin{matrix}\text{the sum of the integrals over } CD\text{, the small circle,}\\ \text{and } EF\text{, when } R\to\infty \text{ and } \epsilon\to 0\end{matrix}\right).$$

It is obvious that the integral over the small circle $\to 0$ when $\epsilon\to 0$.

Also
$$\text{the integral over } CD=i\int\limits_0^\infty e^{-\rho t}\frac{e^{+i\sqrt{(\rho/\kappa)}x}}{h-i\sqrt{(\rho/\kappa)}}\frac{d\rho}{\sqrt{\rho}}.$$

† § 39, (7) and (8).

And $\qquad$ the integral over $EF = i\int\limits_0^\infty e^{-\rho t}\frac{e^{-i\sqrt{(\rho/\kappa)}x}}{h+i\sqrt{(\rho/\kappa)}}\frac{d\rho}{\sqrt{\rho}}.$

Thus

$$F(t) = \frac{1}{\pi}\int\limits_0^\infty e^{-\rho t}\frac{h\cos\sqrt{(\rho/\kappa)}x-\sqrt{(\rho/\kappa)}\sin\sqrt{(\rho/\kappa)}x}{h^2+\rho/\kappa}\frac{d\rho}{\sqrt{\rho}}$$

$$= \frac{2\sqrt{\kappa}}{\pi}\int e^{-\kappa\alpha^2 t}\frac{h\cos\alpha x-\alpha\sin\alpha x}{h^2+\alpha^2}\,d\alpha$$

But $$\int\limits_0 e^{-h\xi}\cos\alpha\xi\,d\xi = \frac{h}{h^2+\alpha^2}.$$

And $$\int\limits_0^\infty e^{-h\xi}\sin\alpha\xi\,d\xi = \frac{\alpha}{h^2+\alpha^2}.$$

Therefore

$$F(t) = \frac{2\sqrt{\kappa}}{\pi}\int\limits_0^\infty e^{-\kappa\alpha^2 t}\left[\int\limits_0^\infty e^{-h\xi}\cos\alpha(x+\xi)\,d\xi\right]d\alpha$$

$$= \frac{2\sqrt{\kappa}}{\pi}\int\limits_0^\infty e^{-h\xi}\left[\int\limits_0^\infty e^{-\kappa\alpha^2 t}\cos\alpha(x+\xi)\,d\alpha\right]d\xi$$

$$= \frac{1}{\sqrt{(\pi t)}}\int\limits_0^\infty e^{-h\xi-(x+\xi)^2/4\kappa t}\,d\xi. \tag{8}$$

Hence, from (6), (7), and (8),

$$v = \frac{2v_0}{\sqrt{\pi}}\int\limits_0^{x/2\sqrt{(\kappa t)}} e^{-u^2}\,du+\frac{v_0}{\sqrt{(\pi\kappa t)}}\int\limits_0^\infty e^{-hu-(x+u)^2/4\kappa t}\,du.$$

**49.** *Instantaneous unit source at $x'$ at $t = 0$ in the semi-infinite solid $x > 0$. Radiation at $x = 0$ into a medium at zero.*

We require a solution of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x > 0,\ t > 0,$$

which shall satisfy

$$-\frac{\partial v}{\partial x}+hv = 0, \quad \text{when } x = 0,\ t > 0,$$

and shall be infinite as $\dfrac{1}{2\sqrt{(\pi\kappa t)}}e^{-(x-x')^2/4\kappa t}$ at $x = x'$, when $t \to 0$, but be zero for every other positive $x$ when $t \to 0$.

Put
$$u = \frac{1}{2\sqrt{(\pi\kappa t)}}e^{-(x-x')^2/4\kappa t}$$
and
$$v = u+w.$$
Then $w$ has to satisfy
$$\frac{\partial w}{\partial t} = \kappa\frac{\partial^2 w}{\partial x^2}, \quad x > 0,\ t > 0,$$
$$w = 0, \quad \text{when } t = 0,\ x > 0,$$
$$-\frac{\partial w}{\partial x}+hw = \frac{\partial u}{\partial x}-hu, \quad \text{when } x = 0,\ t > 0.$$
Write
$$\bar{u} = \int_0^\infty e^{-pt}u\,dt,$$
$$\bar{v} = \int_0^\infty e^{-pt}v\,dt,$$
and
$$\bar{w} = \int^\infty e^{-pt}w\,dt.$$
We know† that
$$\bar{u} = \frac{}{2\sqrt{(\kappa p)}}e^{-\sqrt{(p/\kappa)}|x-x'|}.$$

The equations for $\bar{w}$ are
$$\frac{d^2\bar{w}}{dx^2}-p\bar{w} = 0, \quad x > 0,$$
$$-\frac{d\bar{w}}{dx}+h\bar{w} = \frac{d\bar{u}}{dx}-h\bar{u}, \quad \text{when } x = 0.$$
Thus
$$-\frac{d\bar{w}}{dx}+h\bar{w} = \frac{1}{2\sqrt{(\kappa p)}}\{\sqrt{(p/\kappa)}-h\}e^{-\sqrt{(p/\kappa)}x'}, \quad \text{when } x = 0.$$

† $$\int_0^\infty e^{-a^2x^2-(b^2/x^2)}\,dx = \frac{\sqrt{\pi}}{2|a|}e^{-2|ab|},$$
$$\int_0^\infty e^{-pt-(x^2/4\kappa t)}\frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{p}}\int_0^\infty e^{-t^2-(px^2/4\kappa t^2)}\,dt = \frac{\sqrt{\pi}}{\sqrt{p}}e^{-\sqrt{(p/\kappa)}|x|}.$$

It follows that

$$\bar{w} = \frac{1}{2\sqrt{(\kappa p)}} \frac{\sqrt{(p/\kappa)}-h}{\sqrt{(p/\kappa)}+h} e^{-\sqrt{(p/\kappa)}(x+x')},$$

and

$$\bar{v} = \frac{1}{2\sqrt{(\kappa p)}}\left\{e^{-\sqrt{(p/\kappa)}|x-x'|}+e^{-\sqrt{(p/\kappa)}(x+x')}-\frac{2h}{\sqrt{(p/\kappa)}+h}e^{-\sqrt{(p/\kappa)}(x+x')}\right\}.$$

The first and second terms of $\bar{v}$ are the Laplace Transforms of

$$\frac{1}{2\sqrt{(\pi\kappa t)}}e^{-(x-x')^2/4\kappa t} \quad \text{and} \quad \frac{1}{2\sqrt{(\pi\kappa t)}}e^{-(x+x')^2/4\kappa t}.$$

respectively.

Also we have seen in § 48 (8) that $\frac{}{\sqrt{(p\kappa)}\{\sqrt{(p/\kappa)}+h\}}e^{-\sqrt{(p/\kappa)}(x+x')}$ is the Laplace Transform of

$$\frac{1}{\sqrt{(\pi\kappa t)}}\int\limits_0 e^{-h\xi-(x+x'+\xi)^2/4\kappa t}\,d\xi.$$

Therefore

$$v = \frac{1}{2\sqrt{(\pi\kappa t)}}\left\{e^{-(x-x')^2/4\kappa t}+e^{-(x+x')^2/4\kappa t}-2h\int\limits_0^\infty e^{-h\xi-(x+x'+\xi)^2/4\kappa t}\,d\xi\right\}.$$

**50.** *Unit source at $x'$ at $t = 0$ in $0 < x < l$. The ends $x = 0$ and $x = l$ kept at zero.*

We require a solution of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad 0 < x < l,\, t > 0,$$

which shall vanish when $x = 0$ and $x = l$, when $t > 0$, and shall be infinite as $\frac{1}{2\sqrt{(\pi\kappa t)}}e^{-(x-x')^2/4\kappa t}$ at $x = x'$, when $t \to 0$, but be zero for every other value of $x$ in $0 < x < l$ when $t \to 0$.

As before, write

$$u = \frac{1}{2\sqrt{(\pi\kappa t)}}e^{-(x-x')^2/4\kappa t},$$

and let $$v = u+w.$$

Then $w$ is given by the equations

$$\frac{\partial w}{\partial t} = \kappa\frac{\partial^2 w}{\partial x^2}, \quad 0 < x < l,\, t > 0, \tag{1}$$

$$w = 0, \quad \text{when } t = 0,\ 0 < x < l, \tag{2}$$

$$w = -u, \quad \text{when } x = 0 \text{ and } x = l, \quad t > 0. \tag{3}$$

Let

$$\bar{u} = \int\limits_0^\infty e^{-pt}u\,dt,$$

$$\bar{v} = \int\limits_0^\infty e^{-pt}v\,dt,$$

and

$$\bar{w} = \int\limits_0^\infty e^{-pt}w\,dt.$$

We know that
$$\bar{u} = \frac{1}{2\sqrt{(\kappa p)}}e^{-\sqrt{(p/\kappa)}|x-x'|}.$$

Thus from (1), (2), and (3) the equations for $\bar{w}$ are

$$\kappa\frac{d^2\bar{w}}{dx^2}-p\bar{w} = 0, \quad 0 < x < l,$$

$$\bar{w} = -\frac{1}{2\sqrt{(\kappa p)}}e^{-\sqrt{(p/\kappa)}x'}, \quad \text{when } x = 0,$$

$$= -\frac{1}{2\sqrt{(\kappa p)}}e^{-\sqrt{(p/\kappa)}(l-x')}, \quad \text{when } x = l.$$

Therefore we have

$$\bar{w} = -\frac{1}{2\sqrt{(\kappa p)}}\frac{e^{-\sqrt{(p/\kappa)}(l-x')}\sinh\sqrt{(p/\kappa)}x+e^{-\sqrt{(p/\kappa)}x'}\sinh\sqrt{(p/\kappa)}(l-x)}{\sinh\sqrt{(p/\kappa)}l}$$

$$= -\frac{1}{2\sqrt{(\kappa p)}}\frac{\cosh\sqrt{(p/\kappa)}(l-x-x')-e^{-\sqrt{(p/\kappa)}l}\cosh\sqrt{(p/\kappa)}(x-x')}{\sinh\sqrt{(p/\kappa)}l}.$$

But $\bar{v} = \bar{u}+\bar{w}$.

Therefore

$$\bar{v} = \frac{1}{2\sqrt{(\kappa p)}}\times$$

$$\times\left[e^{-\sqrt{(p/\kappa)}(x-x')}-\frac{\cosh\sqrt{(p/\kappa)}(l-x-x')-e^{-\sqrt{(p/\kappa)}l}\cosh\sqrt{(p/\kappa)}(x-x')}{\sinh\sqrt{(p/\kappa)}l}\right]$$

$$= \frac{1}{2\sqrt{(\kappa p)}}\frac{\cosh\sqrt{(p/\kappa)}(l+x-x')-\cosh\sqrt{(p/\kappa)}(l-x-x')}{\sinh\sqrt{(p/\kappa)}l},$$

when $x < x'$.

And when $x > x'$ we must interchange $x$ and $x'$ in this result.

Then, by the Inversion Theorem,

$$v = \frac{1}{4i\pi\sqrt{\kappa}} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{\cosh\sqrt{(\lambda/\kappa)}(l+x-x') - \cosh\sqrt{(\lambda/\kappa)}(l-x-x')}{\sinh\sqrt{(\lambda/\kappa)}l} \frac{d\lambda}{\sqrt{\lambda}},$$

when $x < x'$.

In this case we use Fig. 10, since the integrand is a single-valued function of $\lambda$ in the region bounded by the closed circuit of this figure, and, as before, we take the radius of $\Gamma$ equal to $\kappa(n+\frac{1}{2})^2(\pi^2/l^2)$, so that it will not pass through a zero of $\sinh\sqrt{(\lambda/\kappa)}l$.

The integral over the circular arcs $\to 0$ when $n \to \infty$ and we find that

$$v = \frac{1}{2\sqrt{\kappa}} \sum_{s=1}^{\infty} e^{-\kappa(s^2\pi^2/l^2)t} \frac{\cos s\pi(l+x-x')/l - \cos s\pi(l-x-x')/l}{[\sqrt{\lambda}(d/d\lambda)\sinh\sqrt{(\lambda/\kappa)}l]_{\lambda=-\kappa(s^2\pi^2/l^2)}}$$

$$= \frac{2}{l} \sum_{1}^{\infty} e^{-\kappa(s^2\pi^2/l^2)t} \sin\frac{s\pi}{l}x \sin\frac{s\pi}{l}x'. \qquad (4)$$

This has been obtained for $x < x'$, but it is symmetrical in $x$ and $x'$ and thus holds for $x > x'$ as well.

For a source of strength $Q$ we replace $2/l$ in (4) by $2Q/l$.

If we take a source of strength $Q = f(x')\,dx'$ at $x'$ at $t = 0$ and integrate the result obtained above, we find

$$v = \frac{2}{l}\int\limits_0^l f(x')\left[\sum_1^{\infty} e^{-\kappa(n^2\pi^2/l^2)t} \sin\frac{n\pi}{l}x \sin\frac{n\pi}{l}x'\right] dx',$$

and with suitable restrictions on $f(x)$ we may write this in the form

$$v = \sum_1^{\infty} e^{-\kappa(n^2\pi^2/l^2)t} a_n \sin\frac{n\pi}{l}x,$$

where $a_n = \frac{2}{l}\int\limits_0^l f(x')\sin\frac{n\pi}{l}x'\,dx'$.

This result may be obtained by the method of § 42. It is, of course, the classical form of the solution obtained from the Fourier's Sine Series for $f(x)$.†

† Cf. *C.H.*, § 20.

**51.** We consider now the *temperature in a wire, along which a steady electrical current is flowing. The ends of the wire are kept at constant temperatures* $v_1$, $v_2$. *Radiation takes place into a medium at a constant temperature* $v_0$. *The initial temperature of the wire is taken as zero.*†

Let the wire be of length $l$, and $K$, $c$, $\rho$, and $H$ its thermal conductivity, specific heat, density, and emissivity. Let $I$ be the strength of the current and $\sigma$ the electrical conductivity, i.e. the reciprocal of the resistance per unit cross-section, per unit length.

Take the element of the wire contained between the sections distant $x$ and $x+dx$ from the end.

The rate of gain of heat in this element from the flow of heat over the sections at $x$ and $x+dx$ is

$$K\omega\frac{\partial^2 v}{\partial x^2}\,dx,$$

$\omega$ being the cross-section of the wire.

The rate at which heat is lost at the surface of the element is

$$H(v-v_0)s\,dx,$$

$s$ being the perimeter of the cross-section and $v_0$ the temperature of the surrounding medium.

The rate of gain of heat due to the current $I$ is

$$\frac{I^2}{\omega\sigma}\,dx.$$

The total rate of gain of heat is therefore

$$\left(K\omega\frac{\partial^2 v}{\partial x^2}-H(v-v_0)s+\frac{I^2}{\omega\sigma}\right)dx.$$

This must be equal to $$\omega c\rho\frac{\partial v}{\partial t}\,dx,$$

and therefore the equation of conduction is

$$\frac{\partial v}{\partial t}=\frac{K}{c\rho}\frac{\partial^2 v}{\partial x^2}-\frac{Hs}{c\rho\omega}(v-v_0)+\frac{I^2}{c\rho\omega^2\sigma}=\kappa\frac{\partial^2 v}{\partial x^2}-bv+a,$$

where $\kappa=\dfrac{K}{c\rho}$, $b=\dfrac{Hs}{c\rho\omega}$, and $a=\dfrac{I^2}{c\rho\omega^2\sigma}+bv_0$.

† Cf. *C.H.*, § 38.

**52.** In the problem stated at the beginning of § 51 we have to determine $v$ from the equations

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2} - bv + a, \quad 0 < x < l,\ t > 0, \tag{1}$$

$$\left.\begin{aligned} v &= v_1, \quad \text{when } x = 0,\\ v &= v_2, \quad \text{when } x = l, \end{aligned}\right\} \quad t > 0, \tag{2}$$

$$v = 0, \quad \text{when } t = 0,\ 0 < x < l. \tag{3}$$

The subsidiary equation is

$$\kappa\frac{d^2\bar{v}}{dx^2} - (b+p)\bar{v} + \frac{a}{p} = 0, \quad 0 < x < l, \tag{4}$$

with $\qquad \bar{v} = \dfrac{v_1}{p}, \quad$ when $x = 0$,

and $\qquad \bar{v} = \dfrac{v_2}{p}, \quad$ when $x = l$. $\qquad$ (5)

Writing $\qquad q = \sqrt{\left(\dfrac{p+b}{\kappa}\right)}, \qquad$ (6)

from (4) and (5) we obtain

$$\bar{v} = \frac{a}{p(p+b)}\left[1 - \frac{\cosh q(\tfrac{1}{2}l - x)}{\cosh \tfrac{1}{2}ql}\right] + \frac{v_2 \sinh qx}{p \sinh ql} + \frac{v_1 \sinh q(l-x)}{p \sinh ql}. \tag{7}$$

We apply the Inversion Theorem to each of the three expressions on the right hand of (7).

From the first we have

$$\frac{a}{2i\pi}\int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}\left[1 - \frac{\cosh \mu(\tfrac{1}{2}l - x)}{\cosh \tfrac{1}{2}\mu l}\right]\frac{d\lambda}{\lambda(\lambda+b)}, \tag{8}$$

where $\mu = \sqrt{\{(\lambda+b)/\kappa\}}$.

The integrand is single-valued inside the closed circuit of Fig. 10 in the $\lambda$-plane, and it has poles at

$$\lambda = 0$$

and $\qquad \lambda = -[\kappa(2n+1)^2\pi^2/l^2 + b], \quad n = 0, 1, 2, \ldots.$

We choose the radius of the circle $\Gamma$ as $b + 4\kappa(n^2\pi^2/l^2)$, so that it does not pass through any of these poles.

When $n \to \infty$, the integral over $\Gamma$ in Fig. 10 vanishes and we obtain for (8)

$$\frac{a}{b}\left[1-\frac{\cosh\sqrt{(b/\kappa)}(\frac{1}{2}l-x)}{\cosh\sqrt{(b/\kappa)}\frac{1}{2}l}\right]- \\ -\frac{4a}{\pi}\sum_{n=0}^{\infty}\frac{e^{-[\kappa(2n+1)^2\pi^2/l^2+b]t}\sin\{(2n+1)\pi/l\}x}{(2n+1)[\kappa(2n+1)^2\pi^2/l^2+b]}. \tag{9}$$

Again from the Inversion Theorem we have for the second expression in (7)

$$\frac{v_2}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\sinh\mu x}{\sinh\mu l}\frac{d\lambda}{\lambda}. \tag{10}$$

We take again Fig. 10, and the poles are at

$$\lambda = 0$$

and $\qquad \lambda = -[\kappa(n^2\pi^2/l^2)+b], \quad n = 1, 2,\ldots.$

In this case we take the radius of $\Gamma$, $(n+\frac{1}{2})^2\pi^2/l^2+b$, and obtain for (10)

$$v_2\left\{\frac{\sinh\sqrt{(b/\kappa)}x}{\sinh\sqrt{(b/\kappa)}l}+\frac{2\kappa\pi}{l^2}\sum_{n=1}^{\infty}\frac{e^{-[\kappa(n^2\pi^2/l^2)+b]t}}{\{\kappa(n^2\pi^2/l^2)+b\}}n\cos n\pi\sin(n\pi/l)x\right\}. \tag{11}$$

Also, from the third expression in (7), we have

$$\frac{v_1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\sinh\mu(l-x)}{\sinh\mu l}\frac{d\lambda}{\lambda}, \tag{12}$$

and this gives, as above,

$$v_1\left\{\frac{\sinh\sqrt{(b/\kappa)}(l-x)}{\sinh\sqrt{(b/\kappa)}l}-\frac{2\kappa\pi}{l^2}\sum_{n=1}^{\infty}\frac{e^{-[\kappa(n^2\pi^2/l^2)+b]t}}{\kappa(n^2\pi^2/l^2)+b}n\sin(n\pi/l)x\right\}. \tag{13}$$

From (9), (11), and (13) we have finally

$$v = \frac{a}{b}\left\{1-\frac{\cosh\sqrt{(b/\kappa)}(\frac{1}{2}l-x)}{\cosh\sqrt{(b/\kappa)}\frac{1}{2}l}\right\}- \\ -\frac{4a}{\pi}\sum_{n=0}^{\infty}\frac{e^{-[\kappa(2n+1)^2\pi^2/l^2+b]t}\sin(2n+1)(\pi/l)x}{(2n+1)[\kappa(2n+1)^2\pi^2/l^2+b]}+ \\ +\frac{v_1\sinh\sqrt{(b/\kappa)}(l-x)+v_2\sinh\sqrt{(b/\kappa)}x}{\sinh\sqrt{(b/\kappa)}l}+ \\ +\frac{2\kappa\pi}{l^2}\sum_{n=1}\frac{e^{-[\kappa(n^2\pi^2/l^2)+b]t}(v_2\cos n\pi-v_1)n\sin(n\pi/l)x}{\kappa(n^2\pi^2/l^2)+b}.$$

## CIRCULAR SYMMETRY

**53.** We now take some problems dealing with conduction in circular cylinders, the temperature distribution being dependent only on the distance from the centre.

*A circular cylinder of radius a has its surface kept at a constant temperature* $v_0$. *The initial temperature through the cylinder is zero. To find the temperature at time t.*

Here the equations for $v$ are

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right), \quad 0 \leqslant r < a,\ t > 0, \tag{1}$$

$$v = 0, \quad \text{when } t = 0,\ 0 \leqslant r < a, \tag{2}$$

$$v = v_0, \quad \text{when } r = a,\ t > 0. \tag{3}$$

The subsidiary equation is

$$\kappa\left(\frac{d^2\bar{v}}{dr^2}+\frac{1}{r}\frac{d\bar{v}}{dr}\right)-p\bar{v} = 0, \quad 0 \leqslant r < a, \tag{4}$$

with
$$\bar{v} = \frac{v_0}{p}, \quad \text{when } r = a. \tag{5}$$

From (4) and (5),† $$\bar{v} = \frac{v_0}{p}\frac{I_0(qr)}{I_0(qa)}, \tag{6}$$
where $q = \sqrt{(p/\kappa)}$.

The problem is thus reduced to finding $v$ from

$$\frac{v_0}{p}\frac{I_0(qr)}{I_0(qa)} = \int\limits_0^\infty e^{-pt}v\,dt. \tag{7}$$

Using the Inversion Theorem,

$$v = \frac{v_0}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{I_0(\mu r)}{I_0(\mu a)}\frac{d\lambda}{\lambda}, \tag{8}$$

where $\mu = \sqrt{(\lambda/\kappa)}$.

Now $I_0(\mu r)/I_0(\mu a)$ is a single-valued function of $\lambda$, so we use Fig. 10. The poles are at the origin, and $\lambda = -\kappa\alpha_1^2, -\kappa\alpha_2^2, \ldots$, where $\pm\alpha_1, \pm\alpha_2, \ldots$ are the roots (all real and simple)‡ of $J_0(\alpha a) = 0$. We take the radius of the circle $\Gamma$ as $\kappa(n+\frac{1}{2})^2\pi^2/a^2$ since in that case we know§ that there is no pole on its circumference.

† See Appendix II, § 5. ‡ *W.B.F.*, § 15.25. § Ibid., § 15.32.

The approximations for $I_0\{\surd(\lambda/\kappa)r\}$ and $I_0\{\surd(\lambda/\kappa)a\}$ show that when $n \to \infty$ the integral over the circle $BCA$ tends to zero.

Thus we can replace $\int\limits_{\gamma-i\infty}^{\gamma+i\infty}$ in (8) by the limit of the integral over the closed circuit of Fig. 10 when $n \to \infty$.

It follows from the Theory of Residues that

$$v = v_0\left\{1+\sum_{1}^{\infty} e^{-\kappa\alpha_s^2 t}\frac{J_0(\alpha_s a)}{[\lambda(d/d\lambda)I_0\{\surd(\lambda/\kappa)a\}]_{\lambda=-\kappa\alpha_s^2}}\right. \tag{9}$$

the pole at the origin giving the first term.

But
$$zI_0'(z) = 2\frac{z^2}{2^2}+4\frac{z^4}{2^2.4^2}+.$$

and
$$\lambda\frac{a}{d\lambda}I_0\{\surd(\lambda/\kappa)a\} = \tfrac{1}{2}\surd(\lambda/\kappa)aI_0'\{\surd(\lambda/\kappa)a\}.$$

Thus
$$v = v_0\left\{1+\frac{2}{a}\sum_{1}^{\infty} e^{-\kappa\alpha_s^2 t}\frac{J_0(\alpha_s a)}{\alpha_s J_0'(\alpha_s a)}\right\}, \tag{10}$$

where $\pm\alpha_1$, $\pm\alpha_2$,... are the roots of $J_0(\alpha a) = 0$.

**54.** *The same as in § 53, but the surface temperature $v_0 \cos\omega t$.*

Here the equations for $\bar{v}$ are

$$\kappa\left(\frac{d^2\bar{v}}{dr^2}+\frac{1}{r}\frac{d\bar{v}}{dr}\right)-p\bar{v} = 0, \quad 0 \leqslant r < a, \tag{1}$$

and
$$\bar{v} = v_0\int^{\infty} e^{-pt}\cos\omega t\, dt, \quad \text{when } r = a,$$

i.e.
$$\bar{v} = \frac{pv_0}{p^2+\omega^2} \quad \text{when } r = a. \tag{2}$$

From (1) and (2),
$$\bar{v} = v_0\frac{I_0(qr)}{I_0(qa)}\frac{p}{p^2+\omega^2}, \tag{3}$$

and, by the Inversion Theorem,

$$v = \frac{v_0}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{I_0(\mu r)}{I_0(\mu a)}\frac{\lambda\, d\lambda}{\lambda^2+\omega^2}, \tag{4}$$

where $\mu = \surd(\lambda/\kappa)$.

We use Fig. 10, as before, and obtain

$$\frac{v}{v_0} = \tfrac{1}{2}e^{i\omega t}\frac{I_0\{\sqrt{(\omega/\kappa)}e^{\frac{1}{4}i\pi}r\}}{I_0\{\sqrt{(\omega/\kappa)}e^{\frac{1}{4}i\pi}a\}}+\tfrac{1}{2}e^{-i\omega t}\frac{I_0\{\sqrt{(\omega/\kappa)}e^{-\frac{1}{4}i\pi}r\}}{I_0\{\sqrt{(\omega/\kappa)}e^{-\frac{1}{4}i\pi}a\}}+ \\ +\frac{2}{a}\sum_{s=1}^{\infty} e^{-\kappa\alpha_s^2 t}\frac{J_0(\alpha_s r)}{\alpha_s J_0'(\alpha_s a)}\frac{\kappa^2\alpha_s^4}{\kappa^2\alpha_s^4+\omega^2}. \tag{5}$$

Using the notation†

$$\begin{aligned} \operatorname{ber} z+i\operatorname{bei} z &= I_0(ze^{\frac{1}{4}i\pi}), \\ \operatorname{ber} z-i\operatorname{bei} z &= I_0(ze^{-\frac{1}{4}i\pi}), \end{aligned} \tag{6}$$

this reduces to

$$= \cos\omega t\frac{\operatorname{ber}\sqrt{(\omega/\kappa)}r\operatorname{ber}\sqrt{(\omega/\kappa)}a+\operatorname{bei}\sqrt{(\omega/\kappa)}r\operatorname{bei}\sqrt{(\omega/\kappa)}a}{\operatorname{ber}^2\sqrt{(\omega/\kappa)}a+\operatorname{bei}^2\sqrt{(\omega/\kappa)}a}+ \\ +\sin\omega t\frac{\operatorname{ber}\sqrt{(\omega/\kappa)}r\operatorname{bei}\sqrt{(\omega/\kappa)}a-\operatorname{bei}\sqrt{(\omega/\kappa)}r\operatorname{ber}\sqrt{(\omega/\kappa)}a}{\operatorname{ber}^2\sqrt{(\omega/\kappa)}a+\operatorname{bei}^2\sqrt{(\omega/\kappa)}a}+ \\ +\frac{2\kappa^2}{a}\sum_{1}^{\infty} e^{-\kappa\alpha_s^2 t}\frac{J_0(\alpha_s r)}{J_0'(\alpha_s a)}\frac{\alpha_s^3}{\kappa^2\alpha_s^4+\omega^2}, \tag{7}$$

where $\pm\alpha_1$, $\pm\alpha_2$, ... are the roots of $J_0(\alpha a) = 0$.

**55.** *Instantaneous Cylindrical Surface Source of strength $Q$ over $r = r'$ at $t = 0$. The surface $r = a$ kept at zero.*

We have seen in § 46 that the temperature due to an Instantaneous Cylindrical Surface Source of strength $Q$ over $r = r'$ at $t = 0$ in the infinite solid is given by

$$\frac{Q}{4\pi\kappa t}e^{-(r^2+r'^2)/4\kappa t}I_0\left(\frac{rr'}{2\kappa t}\right)$$

or

$$\frac{Q}{2\pi}\int\limits^{\infty} e^{-\kappa\alpha^2 t}\alpha J_0(\alpha r)J_0(\alpha r')\,d\alpha,$$

by Weber's Second Integral.‡

Thus, for the problem in this section, we have to solve the equations

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right), \quad 0 \leqslant r < a,\ t > 0, \tag{1}$$

$$v = 0, \quad \text{when } r = a,\ t > 0, \tag{2}$$

and

$$v = u+w, \tag{3}$$

† *W.B.F.*, p. 81. ‡ Ibid., § 13.31.

where $$u = \frac{Q}{2\pi}\int\limits_0^\infty e^{-\kappa\alpha^2 t}\alpha J_0(\alpha r)J_0(\alpha r')\,d\alpha, \tag{4}$$

and $$\frac{\partial w}{\partial t} = \kappa\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right), \quad 0 \leqslant r < a,\ t > 0, \tag{5}$$

$$w = 0, \quad \text{when } t = 0,\ 0 \leqslant r < a, \tag{6}$$

$$w = -u, \quad \text{when } r = a,\ t > 0. \tag{7}$$

Writing as usual $\bar{u} = \int\limits_0^\infty e^{-pt}u\,dt$, etc., and $q = \sqrt{(p/\kappa)}$, we know† that

$$\left.\begin{aligned}\bar{u} &= \frac{Q}{2\pi\kappa}I_0(qr')K_0(qr), \quad \text{when } r > r',\\ &= \frac{Q}{2\pi\kappa}I_0(qr)K_0(qr'), \quad \text{when } r < r'.\end{aligned}\right\} \tag{8}$$

† One way of establishing this result is as follows: it is known (*G. and M.*, p. 74 (59); *W.B.F.*, § 11.3 (8)) that

$$K_0(qR) = I_0(qr')K_0(qr)+2\sum_1^\infty I_n(qr')K_n(qr)\cos n\theta,$$

where $R = \sqrt{(r^2+r'^2-2rr'\cos\theta)}$ and $r > r'$.

Also (*G. and M.*, p. 51 (33); *W.B.F.*, § 6.22 (15))

$$K_0(qR) = \tfrac{1}{2}\int\limits^\infty e^{-pt}\frac{e^{-R^2/(4\kappa t)}}{t}\,dt.$$

Therefore $$\int\limits_0 K_0(qR)\,d\theta = 2\pi I_0(qr')K_0(qr), \quad r > r',$$

i.e. $$\frac{1}{2}\int\limits_0^{2\pi}\left[\int\limits_0^\infty e^{-pt}\frac{e^{-R^2/(4\kappa t)}}{t}\,dt\right]d\theta = 2\pi I_0(qr')K_0(qr).$$

The integral converges uniformly and we can integrate under the integral sign, so that we have

$$\frac{1}{2}\int\limits_0^\infty e^{-pt}\frac{e^{-(r^2+r'^2)/4\kappa t}}{t}\left[\int\limits_0^{2\pi} e^{(rr'/2\kappa t)\cos\theta}\,d\theta\right]dt = 2\pi I_0(qr')K_0(qr).$$

Thus $$\pi\int\limits_0^\infty e^{-pt}\frac{e^{-(r^2+r'^2)/4\kappa t}}{t}I_0\left(\frac{rr'}{2\kappa t}\right)dt = 2\pi I_0(qr')K_0(qr),$$

i.e. $$2\kappa\pi\int\limits_0^\infty e^{-pt}\left[\int\limits_0^\infty e^{-\kappa\alpha^2 t}\alpha J_0(\alpha r)J_0(\alpha r')\,d\alpha\right]dt = 2\pi I_0(qr')K_0(qr).$$

Hence $$\int\limits_0^\infty e^{-pt}\left[\int\limits_0^\infty e^{-\kappa\alpha^2 t}\alpha J_0(\alpha r)J_0(\alpha r')\,d\alpha\right]dt = \frac{1}{\kappa}I_0(qr')K_0(qr),$$

when $r > r'$.

Interchange $r$ and $r'$ for $r < r'$.

Thus the equations for $\bar{w}$ are

$$\kappa\left(\frac{d^2\bar{w}}{dr^2}+\frac{1}{r}\frac{d\bar{w}}{dr}\right)-p\bar{w} = 0, \quad 0 \leqslant r < a, \tag{9}$$

and
$$\bar{w} = -\frac{Q}{2\pi\kappa}I_0(qr')K_0(qa), \quad \text{when } r = a. \tag{10}$$

From (9) and (10),

$$\bar{w} = -\frac{Q}{2\pi\kappa}\frac{I_0(qr')K_0(qa)}{I_0(qa)}I_0(qr). \tag{11}$$

And

$$\bar{v} = \bar{u}+\bar{w}$$
$$= \frac{Q}{2\pi\kappa}\frac{I_0(qr')}{I_0(qa)}\{I_0(qa)K_0(qr)-I_0(qr)K_0(qa)\}, \quad r > r', \tag{12}$$
$$= \frac{Q}{2\pi\kappa}\frac{I_0(qr)}{I_0(qa)}\{I_0(qa)K_0(qr')-I_0(qr')K_0(qa)\}, \quad r < r'. \tag{13}$$

Using the Inversion Formula, and writing $\mu = \sqrt{(\lambda/\kappa)}$,

$$v = \frac{Q}{4\pi^2 i\kappa}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{I_0(\mu r')}{I_0(\mu a)}[I_0(\mu a)K_0(\mu r)-I_0(\mu r)K_0(\mu a)]\,d\lambda, \tag{14}$$

when $r > r'$, and we have to interchange $r$ and $r'$ when $r < r'$.

The integrand in (14) is a single-valued function of $\lambda$ and we again use Fig. 10, taking the radius of the circle $\Gamma$ as $\kappa(n+\frac{1}{2})^2(\pi^2/a^2)$ so that it does not pass through a pole. When $n \to \infty$, the integral over $\Gamma$ in the figure tends to zero, and we can replace $\int\limits_{\gamma-i\infty}^{\gamma+i\infty}$ in (14) by the integral over the closed circuit.

It follows that, when $r > r'$,

$$v = \frac{Q}{2\pi\kappa}\sum_{1}^{\infty} e^{-\kappa\alpha_s^2 t}\frac{I_0(i\alpha_s r')[-I_0(i\alpha_s r)K_0(i\alpha_s a)]}{[(d/d\lambda)I_0\{\sqrt{(\lambda/\kappa)}a\}]_{\lambda=-\kappa\alpha^2}}, \tag{15}$$

the summation being over the positive roots $\alpha_1, \alpha_2,\ldots$ of $J_0(\alpha a) = 0$.

But
$$\frac{d}{d\lambda}I_0\{\sqrt{(\lambda/\kappa)}a\} = \frac{a}{2\sqrt{(\lambda\kappa)}}I_0'\{\sqrt{(\lambda/\kappa)}a\}.$$

And $$I_0'(z)K_0(z)-I_0(z)K_0'(z) = 1/z.$$

Hence we have

$$v = \frac{Q}{\pi a^2}\sum_1^\infty e^{-\kappa\alpha_s^2 t}\frac{J_0(\alpha_s r)J_0(\alpha_s r')}{[J_0'(\alpha_s a)]^2}, \tag{16}$$

and, as this is symmetrical in $r$ and $r'$, it holds both for $r > r'$ and $r < r'$.

Put $Q = 2\pi r' f(r')\,dr'$ and integrate from 0 to $a$ in (16).

In this way we obtain the solution for an initial temperature $f(r)$ in the cylinder, the surface being kept at zero, namely,

$$v = \frac{2}{a^2}\int_0^a r'f(r')\left[\sum_1^\infty e^{-\kappa\alpha_s^2 t}\frac{J_0(\alpha_s r)J_0(\alpha_s r')}{[J_0'(\alpha_s a)]^2}\right]dr'.$$

With suitable restrictions on $f(r)$ we may invert the order of integration and summation and obtain

$$v = \frac{2}{a^2}\sum_1^\infty e^{-\kappa\alpha_s^2 t}\frac{J_0(\alpha_s r)}{[J_0'(\alpha_s a)]^2}\int_0^a r'f(r')J_0(\alpha_s r')\,dr'.$$

**56.** *When radiation takes place at $r = a$ into a medium at temperature zero*, we replace (2) of § 55 by

$$\frac{\partial v}{\partial r}+hv = 0, \quad \text{when } r = a.$$

Then, with the same notation as in § 55, we have

$$\bar{w} = -\frac{Q}{2\pi\kappa}I_0(qr')\frac{qK_0'(qa)+hK_0(qa)}{qI_0'(qa)+hI_0(qa)}I_0(qr).$$

But $\bar{v} = \bar{u}+\bar{w}$.

Thus

$$\bar{v} = \frac{Q}{2\pi\kappa}\frac{K_0(qr)[qI_0'(qa)+hI_0(qa)]-I_0(qr)[qK_0'(qa)+hK_0(qa)]}{qI_0'(qa)+hI_0(qa)}\times$$
$$\times I_0(qr'), \quad r > r',$$
$$= \frac{Q}{2\pi\kappa}\frac{K_0(qr')[qI_0'(qa)+hI_0(qa)]-I_0(qr')[qK_0'(qa)+hK_0(qa)]}{qI_0'(qa)+hI_0(qa)}\times$$
$$\times I_0(qr), \quad r < r'.$$

Using the Inversion Formula, and proceeding as in § 55, we find

$$v = \frac{Q}{\pi a^2}\sum_1^\infty e^{-\kappa\alpha_s^2 t}\frac{\alpha_s^2 J_0(\alpha_s r)J_0(\alpha_s r')}{(h^2+\alpha_s^2)J_0^2(\alpha_s a)}, \quad r \lesseqgtr r',$$

the summation being taken over the positive roots† of

$$\alpha J_0'(\alpha a)+hJ_0(\alpha a) = 0.$$

Putting $Q = 2\pi r'f(r')\,dr'$, we obtain the corresponding solution for an initial temperature $f(r)$ in the cylinder, in the form

$$v = \frac{2}{a^2}\sum_{s=1}^{\infty} e^{-\kappa\alpha_s^2 t}\frac{\alpha_s^2 J_0(\alpha_s r)}{(h^2+\alpha_s^2)J_0^2(\alpha_s a)}\int_0^a r'f(r')J_0(\alpha_s r')\,dr'.$$

If we put $h = \infty$, this gives the result of § 55.

If we put $h = 0$, we have the case where no heat escapes at the surface. It will be noticed that in this case we have in the summation a term corresponding to $\alpha = 0$ as well as the positive roots $\alpha_1, \alpha_2,\ldots$ of $J_0'(\alpha a) = 0$.

## Spherical Symmetry

**57.** *Sphere of radius a. Initial temperature constant, $v_0$. Radiation at surface into medium at zero.*

The equations for $v$ are

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}\right), \quad 0 \leqslant r < a,\ t > 0, \tag{1}$$

$$v = v_0, \quad \text{when } t = 0,\ 0 \leqslant r < a, \tag{2}$$

$$\frac{\partial v}{\partial r}+hv = 0, \quad \text{when } r = a,\ t > 0. \tag{3}$$

The subsidiary equation is

$$\kappa\left(\frac{d^2\bar v}{dr^2}+\frac{2}{r}\frac{d\bar v}{dr}\right)-p\bar v = -v_0, \quad 0 \leqslant r < a, \tag{4}$$

with

$$\frac{d\bar v}{dr}+h\bar v = 0, \quad \text{when } r = a. \tag{5}$$

These may be written

$$\frac{d^2}{dr^2}(\bar v r)-q^2(\bar v r) = -\frac{v_0 r}{\kappa}, \quad 0 \leqslant r < a,$$

and

$$\frac{d}{dr}(\bar v r)+\frac{ah-1}{a}(\bar v r) = 0, \quad \text{when } r = a. \tag{6}$$

† All real and simple: *W.B.F.*, §§ 15.23, 15.25.

Solving (6), using $(\bar{v}r) = 0$ when $r = 0$, we have

$$\bar{v} = \frac{v_0}{p} - \frac{ha^2v_0}{rp} \frac{\sinh qr}{aq\cosh qa + (ah-1)\sinh qa}, \tag{7}$$

where $q = \sqrt{(p/\kappa)}$.

We use the Inversion Theorem to find the part of $v$ corresponding to the second term on the right hand of (7).

This gives

$$\frac{ha^2v_0}{2i\pi r} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{\sinh \mu r}{(ah-1)\sinh \mu a + \mu a \cosh \mu a} \frac{d\lambda}{\lambda} \tag{8}$$

where, as usual, $\mu = \sqrt{(\lambda/\kappa)}$.

We know† that the roots $\pm\xi_1, \pm\xi_2,\ldots$ of

$$(ah-1)\sin\xi + \xi\cos\xi = 0, \tag{9}$$

are all real, and their values can be obtained graphically by drawing the curves

$$y = \tan x \quad \text{and} \quad y = -\frac{x}{ah-1}$$

on the same diagram.

Thus the roots of

$$(ah-1)\sinh \mu a + \mu a \cosh \mu a = 0 \tag{10}$$

are given by $\qquad \mu a = \pm i\xi_1, \pm i\xi_2$, etc.,

and the poles of the integrand in (8), which is single-valued in $\lambda$, are at

$$\lambda = 0 \quad \text{and} \quad \lambda = -\kappa\frac{\xi_1^2}{a^2}, \ -\kappa\frac{\xi_2^2}{a^2}, \ \ldots.$$

We take again Fig. 10 and choose the radius of the circle $\Gamma$ as $\kappa(n^2\pi^2/a^2)$, so that it will not pass through a pole.

It will be found that when $n \to \infty$ the integral over the part of the circle $\Gamma$ in Fig. 10 tends to zero, so we can replace $\int_{\gamma-i\infty}^{\gamma+i\infty}$ by the limit of the integral over this closed circuit. Then, using the Theory of Residues, we obtain the value of (8) as an infinite series.

† *C.H.*, p. 137.

The pole at $\lambda = 0$ gives, at once, $v_0$.

The pole at $\lambda = -\kappa(\xi_1^2/a^2)$ gives

$$\frac{ha^2 v_0}{} e^{-\kappa(\xi_1^2/a^2)t} \frac{\sinh i(\xi_1 r/a)}{[(d/d\lambda)\{\lambda(ah-1)\sinh\mu a+\lambda\mu a\cosh\mu a\}]_{\lambda=-\kappa(\xi_1^2/a^2)}}.$$

But

$$\frac{a}{d\lambda}\lambda[(ah-1)\sinh\mu a+\mu a\cosh\mu a]$$

$$= a^2[h\cosh\mu a+\mu\sinh\mu a]\lambda\frac{d\mu}{d\lambda}, \quad \text{when } \lambda = -\kappa\frac{\xi^2}{a^2},$$

$$= \tfrac{1}{2}a^2\cosh\mu a(h+\mu\tanh\mu a)\sqrt{(\lambda/\kappa)}$$

$$= i\xi\cos\xi\frac{\{\xi^2+ah(ah-1)\}}{ah-1}.$$

Taking account of the position of $\xi_1$, $\xi_2$,..., we find that the pole at $\xi_n$ gives

$$(-1)^n\frac{2ha^2v_0}{r}e^{-\kappa(\xi_n^2/a^2)t}\frac{\sqrt{\{\xi_n^2+(ah-1)^2\}}}{\xi_n^2+ah(ah-1)}\frac{\sin(r\xi_n/a)}{\xi_n}.$$

Thus

$$v = \frac{2ha^2v_0}{r}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{\{\xi_n^2+(ah-1)^2\}}}{\xi_n^2+ah(ah-1)}\frac{\sin(r\xi_n/a)}{\xi_n}e^{-\kappa(\xi_n^2/a^2)t}. \tag{11}$$

**58.** The examples in these sections seem sufficient illustration of the use of this method in solving problems in Conduction of Heat. Questions involving a solid bounded by two concentric cylinders, or a solid bounded internally by a cylinder, and more complicated problems dealing with a sphere, can be treated in the same way.†

It may be noticed that no attempt has been made above to verify that the expressions found in these sections do, in fact, satisfy the given differential equations and the initial and boundary conditions.

In some, direct verification of the final result offers no difficulty. In others, the simplest verification is given by returning to the solution obtained in the form of an integral of type $\int_{\gamma-i\infty}^{\gamma+i\infty}$.

† Carslaw and Jaeger, *Phil. Mag.* (7), **26** (1938), 473.

In many problems we can choose a new path $L'$, lying as in Fig. 15, which begins at infinity in the direction $\arg\lambda = -\beta$, where $\pi > \beta > \frac{1}{2}\pi$, passes to the right of the origin, keeping

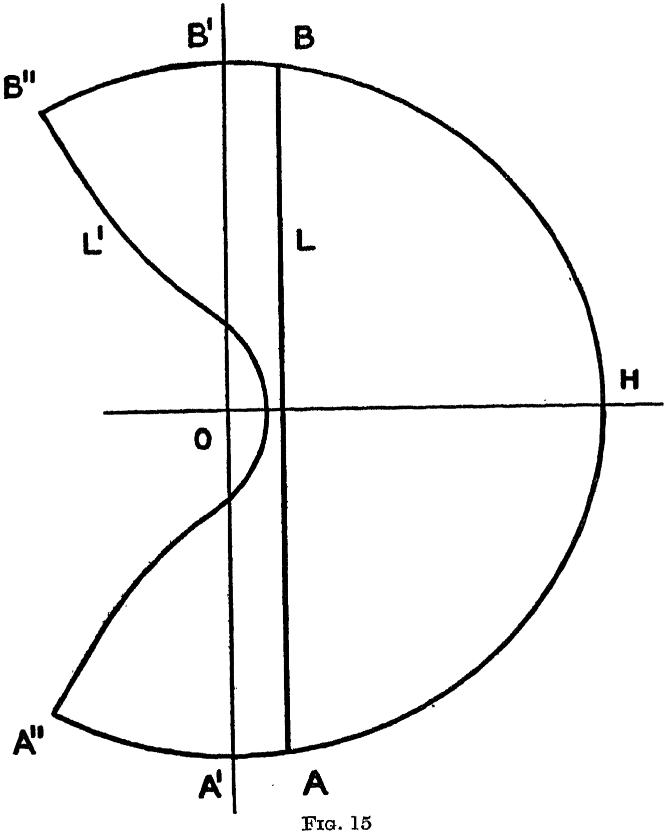


FIG. 15

all singularities of the integrand to the left, and ends in the direction $\arg\lambda = \beta$. Then, owing to the presence of the term $e^{\lambda t}$, and the fact that no singularities of the integrand lie between the paths $L$ $(\gamma - i\infty, \gamma + i\infty)$ and $L'$, we are able to transform the path $L$ into $L'$, since the integral over the part of the circle intercepted between $L$ and $L'$ tends to zero as the radius tends to infinity.

Now on the path $L'$ the integral will be found to be uniformly convergent in $x$ (or $r$) in the given region, when $t$ is a fixed positive number, and uniformly convergent in $t$ for $t \geqslant 0$, when $x$ (or $r$) is fixed. We can then differentiate under the sign of integration and we see that the given differential equation is satisfied. For the same reason the initial and boundary conditions will be found to be satisfied.

As an example we verify that the solution given in § 40 (9),

$$v = v_0 + \frac{v_1 - v_0}{2i\pi} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}} \frac{d\lambda}{\lambda}, \tag{1}$$

satisfies the differential equation, § 40 (1),

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}, \quad 0 < x < l,\ t > 0, \tag{2}$$

and the boundary and initial conditions, § 40 (2), (3), and (4).

These, written in full, are:†

$$\text{for fixed } t > 0, \quad \partial v/\partial x \to 0, \quad \text{as } x \to 0, \tag{3}$$

$$\text{for fixed } t > 0, \quad v \to v_1, \quad \text{as } x \to l, \tag{4}$$

$$\text{for fixed } x \text{ in } 0 < x < l, \quad v \to v_0, \quad \text{as } t \to 0. \tag{5}$$

I. First we need some results as to the order of magnitude of the expression $\dfrac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}$ in the integral of (1), and its first and second differential coefficients with respect to $x$.

Since

$$|\cosh(a+ib)| = (\cosh^2 a - \sin^2 b)^{\frac{1}{2}} = (\sinh^2 a + \cos^2 b)^{\frac{1}{2}},$$

$$\cosh a > |\cosh(a+ib)| > \sinh a, \quad \text{when } a > 0.$$

Let $$\lambda = \kappa R e^{i\theta}, \quad \pi > \theta_0 \geqslant \theta \geqslant 0 \text{ and } 0 < x < l. \tag{6}$$

Then $$|\cosh x\sqrt{(\lambda/\kappa)}| < \cosh(xR^{\frac{1}{2}} \cos \tfrac{1}{2}\theta)$$

and $$|\cosh l\sqrt{(\lambda/\kappa)}| > \sinh(lR^{\frac{1}{2}} \cos \tfrac{1}{2}\theta).$$

Therefore

$$\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}} < e^{-(l-x)R^{\frac{1}{2}}\cos\frac{1}{2}\theta} \left(\frac{1+e^{-2xR^{\frac{1}{2}}\cos\frac{1}{2}\theta}}{1-e^{-2lR^{\frac{1}{2}}\cos\frac{1}{2}\theta}}\right).$$

But $$1 + e^{-2xR^{\frac{1}{2}}\cos\frac{1}{2}\theta} < 2$$

and $$1 - e^{-2lR^{\frac{1}{2}}\cos\frac{1}{2}\theta} > 1 - e^{-2lR^{\frac{1}{2}}\cos\frac{1}{2}\theta_0}$$

$$> \tfrac{1}{2}, \quad \text{when } R > R_0, \quad \text{say}.$$

† Boundary and initial conditions in such cases are always to be interpreted in this way.

Hence $$\left|\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\right| < 4e^{-(l-x)R^{\frac{1}{2}}\cos\frac{1}{2}\theta}, \quad \text{when } R > R_0. \tag{7}$$

Similarly, and subject to the conditions in (6),

$$\left|\sqrt{\left(\frac{\lambda}{\kappa}\right)}\frac{\sinh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\right| < 4R^{\frac{1}{2}}e^{-(l-x)R^{\frac{1}{2}}\cos\frac{1}{2}\theta}, \quad \text{when } R > R_0,$$

and $$\left|\frac{\lambda}{\kappa}\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\right| < 4Re^{-(l-x)R^{\frac{1}{2}}\cos\frac{1}{2}\theta}, \quad \text{when } R > R_0. \tag{8}$$

These results hold also for $x = 0$ and $x = l$.

II. We shall now show that the path $(\gamma - i\infty, \gamma + i\infty)$ of the integral in (1) can be replaced by the path $L'$ of Fig. 15, when $t > 0$ and $0 < x < l$.

To prove this we have to show that

$$\int e^{\lambda t}\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda}$$

taken over the arcs $BB'B''$ and $A''A'A$ of the circle of radius $\kappa R$ in Fig. 15 tends to zero as $R \to \infty$.

Let $I_1$ and $I_2$ be these integrals over $BB'$ and $B'B''$. Then from (7), replacing $\theta_0$ of (6) by $\beta > \frac{1}{2}\pi$, we have

$$|I_1| < 4e^{\gamma t}\sin^{-1}\left(\frac{\gamma}{\kappa R}\right), \quad \text{when } R > R_0,$$

and $$I_1 \to 0, \quad \text{when } R \to \infty.$$

Also $$\begin{aligned}|I_2| &< 4\int\limits_{\frac{1}{2}\pi}^{\beta} e^{\kappa Rt\cos\theta}\,d\theta \\ &< 4\int\limits_{0}^{\beta-\frac{1}{2}\pi} e^{-\kappa Rt\sin\theta}\,d\theta \\ &< 4\int\limits_{0}^{\frac{1}{2}\pi} e^{-2\kappa Rt\theta/\pi}\,d\theta \\ &< \frac{2\pi}{\kappa Rt}.\end{aligned}$$

Thus $I_2 \to 0$, when $R \to \infty$.

A similar argument applies to the integrals over $AA'$ and $A'A''$. Thus we can write our solution (1) as

$$v = v_0 + \frac{v_1 - v_0}{2i\pi}\int\limits_{L'} e^{\lambda t}\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda}. \tag{9}$$

III. Using (7) and (8) it is easy to show that we have the uniform convergence which allows differentiation under the integral sign in (9) for $\partial v/\partial t$ and $\partial^2 v/\partial x^2$, when $t > 0$ and $0 < x < l$.

It follows at once that $v$, as given by (9), and therefore also by (8), satisfies the differential equation (2), when $t > 0$ and $0 < x < l$.

IV. We now show that it satisfies the boundary conditions (3) and (4). We know that

$$\frac{\partial v}{\partial x} = \frac{v_1-v_0}{2i\pi\kappa^{\frac{1}{2}}}\int_{L'} e^{\lambda t}\frac{\sinh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\sqrt{\lambda}} \quad \text{in} \quad 0 < x < l. \tag{10}$$

From (8) we see that this integral converges uniformly for a fixed $t > 0$ in $0 \leqslant x \leqslant l$.

It is thus a continuous function of $x$ in this interval.

But it is zero when $x = 0$. Therefore

$$\lim_{x\to 0}\frac{\partial v}{\partial x} = 0 \quad \text{for fixed } t > 0.$$

Again
$$\int_{L'} e^{\lambda t}\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda}$$

converges uniformly in $0 \leqslant x \leqslant l$ for a fixed $t > 0$.

Thus
$$\lim_{x\to l} v = v_0+\frac{v_1-v_0}{2i\pi}\int e^{\lambda t}\frac{d\lambda}{\lambda}$$
$$= v_1,$$

since
$$\frac{1}{2i\pi}\int_{L'} e^{\lambda t}\frac{d\lambda}{\lambda} = 1, \quad \text{when } t > 0.$$

V. Lastly we show that the initial condition (5) is satisfied.

For fixed $x$ in $0 < x < l$ the integral in (9) over $L'$ is uniformly convergent in $t \geqslant 0$, and it is thus a continuous function of $t$ in $t \geqslant 0$.

Hence
$$\lim_{t\to 0} v = v_0+\frac{v_1-v_0}{2i\pi}\int\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda}.$$

But, at every point of the arc $B''B'HAA''$ of Fig. 15,

$$\left|\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\right| \quad 4e^{-(l-x)R^{\frac{1}{2}}\cos\frac{1}{2}\beta}.$$

It follows that

$$\int\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda} \quad \text{over } B''BHAA''$$

tends to zero when $R \to \infty$.

Hence
$$\int_{L'}\frac{\cosh x\sqrt{(\lambda/\kappa)}}{\cosh l\sqrt{(\lambda/\kappa)}}\frac{d\lambda}{\lambda} = 0,$$

since there are no poles within the closed circuit of Fig. 15.

Therefore
$$\lim_{t\to 0} v = v_0, \quad \text{when } 0 < x < l.$$

This method of transforming $L$ into $L'$ and then verifying the result for $L'$ is suitable for problems in Conduction of Heat and some in other fields. But cases arise in the theory of

vibrations and in electrical theory in which the transform $\bar{v}(\lambda)$ has a sequence of poles extending to infinity either along the imaginary axis or along a line parallel to it. In such cases the path $L$ cannot be transformed into $L'$ and the method is not available. The most satisfactory procedure then may be to perform the verification in much the same way† on the integral over the path $L$; this can usually be done if the initial and boundary conditions are sufficiently well-behaved functions of the space and time variables respectively.‡

† The reader who is interested in these questions should consult a series of papers by Churchill in the *Math. Annalen* (see p. 91, above).

‡ Cf. Jeffreys, *Operational Methods in Mathematical Physics*, C.U.P., § 8.7.

## CHAPTER VII

## VIBRATIONS OF CONTINUOUS MECHANICAL SYSTEMS

**59.** *Longitudinal vibrations of a uniform bar.*

Let $X$ be the stress at the point $x$ of the bar, $u$ the displacement at that point, $\rho$ the density of the bar, $F$ the body force per unit mass, and $E$ Young's modulus, then

$$X = E\frac{\partial u}{\partial x}. \tag{1}$$

The equation of motion is

$$\frac{\partial X}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} - \rho F, \tag{2}$$

or, using (1),
$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\frac{F}{c^2}, \tag{3}$$

where
$$c^2 = E/\rho. \tag{4}$$

**60.** *A bar of length $l$ is at rest in its equilibrium position with the end $x = 0$ fixed. A constant force $S$ per unit area is applied at $t = 0$ at the free end.*

We have to solve

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < l,\ t > 0,$$

with
$$u = 0,\ \frac{\partial u}{\partial t} = 0, \quad \text{when } t = 0,\ 0 < x < l,$$

$$u = 0, \quad \text{when } x = 0,\ t > 0,$$

$$\frac{\partial u}{\partial x} = \frac{S}{E}, \quad \text{when } x = l,\ t > 0.$$

The subsidiary equation is

$$\frac{d^2\bar{u}}{dx^2} - \frac{p^2}{c^2}\bar{u} = 0,$$

to be solved with

$$\bar{u} = 0, \quad \text{when } x = 0,$$

and
$$\frac{d\bar{u}}{dx} = \frac{S}{Ep}, \quad \text{when } x = l.$$

The solution is $$\bar{u} = \frac{Sc}{Ep^2}\frac{\sinh px/c}{\cosh pl/c}.$$

Therefore, by the Inversion Theorem,

$$u = \frac{Sc}{2\pi iE}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda^2}\frac{\sinh \lambda x/c}{\cosh \lambda l/c}\, d\lambda. \tag{1}$$

The integrand is a single-valued function of $\lambda$ with a simple pole at $\lambda = 0$ and simple poles at $\lambda = \pm\frac{(2n+1)i\pi c}{2l}$, $n = 0, 1, 2,\ldots$.

We use the contour of Fig. 10, choosing $R = n\pi c/l$ so that $\Gamma$ does not pass through any pole of the integrand. It will be found that the integral over $BCA$ tends to zero when $n \to \infty$. Thus the integral in (1) equals $2i\pi$ times the sum of the residues at the poles of its integrand.

The residue at $\lambda = 0$ is $x/c$, and that at $\lambda = (2n+1)i\pi c/2l$ is

$$-\frac{4l(-1)^n}{\pi^2 c(2n+1)^2} e^{(2n+1)\pi ict/2l} \sin\frac{(2n+1)\pi x}{2l}.$$

Therefore the solution is

$$u = \frac{Sx}{E} - \frac{8Sl}{\pi^2 E}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l}\cos\frac{(2n+1)\pi ct}{2l}.$$

**61.** *Vibrations of a bar under its own weight.*

The bar is hung vertically and clamped so that the displacement is zero at all points. At $t = 0$ it is released save at the upper point.

By § 59 (3) we have to solve

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\frac{g}{c^2}, \quad t > 0,\ 0 < x < l,$$

with $$u = \frac{\partial u}{\partial t} = 0, \quad \text{when } t = 0,$$

and $$u = 0, \quad \text{when } x = 0,\ t > 0,$$

$$\frac{\partial u}{\partial x} = 0, \quad \text{when } x = l,\ t > 0.$$

The subsidiary equation is

$$\frac{d^2\bar{u}}{dx^2}-\frac{p^2}{c^2}\bar{u} = -\frac{g}{c^2p},$$

to be solved with

$$\bar{u} = 0, \quad \text{when } x = 0,$$

$$\frac{d\bar{u}}{dx} = 0, \quad \text{when } x = l.$$

The solution is

$$\bar{u} = \frac{g}{p^3}-\frac{g}{p^3}\frac{\cosh p(l-x)/c}{\cosh pl/c}.$$

Therefore, using the Inversion Theorem for the second term,

$$u = \tfrac{1}{2}gt^2-\frac{g}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\cosh\lambda(l-x)/c}{\cosh\lambda l/c}\,\frac{d\lambda}{\lambda^3}. \tag{1}$$

The integrand is a single-valued function of $\lambda$ having a triple pole at $\lambda = 0$ with residue $\dfrac{t^2}{2}+\dfrac{x(x-2l)}{2c^2}$, and simple poles at $\lambda = \pm(2n+1)\pi ci/2l$, $n = 0, 1, 2,\ldots$, with residues

$$\frac{8l^2(-1)^n}{(2n+1)^3\pi^3c^2}e^{\pm(2n+1)i\pi ct/2l}\cos\frac{(2n+1)\pi(l-x)}{2l}.$$

Thus, using the contour of Fig. 10, we have in the usual way

$$u = \frac{gx(2l-x)}{2c^2}-\frac{16gl^2}{\pi^3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\frac{(2n+1)\pi(l-x)}{2l}\cos\frac{(2n+1)\pi ct}{2l}.$$

**62.** *Bar of length $l$ and area $a$. The end $x = 0$ is fixed, a mass $m$ is attached to the end $x = l$. The bar is initially stretched by a tension $S$ per unit area, and at $t = 0$ the end $x = l$ is released.*†

Let $u$ be the displacement of the point $x$ of the bar, and $\xi$ that of the mass $m$, so that $\xi = \lim_{x\to l} u$.

At $t = 0$ we have

$$u = \frac{Sx}{E}, \qquad \xi = \frac{Sl}{E}.$$

† Timoshenko, *Vibration Problems in Engineering* (1928), p. 211.

The equation of motion of the bar is

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0, \quad 0<x<l,\, t>0,$$

$$\text{with} \quad u=0, \quad \text{when } x=0,\, t>0, \tag{1}$$

$$\text{and} \quad u=\frac{Sx}{E},\ \frac{\partial u}{\partial t}=0, \quad \text{when } t=0,\, 0<x<l.$$

The equation of motion of the mass $m$ is

$$m\frac{d^2\xi}{dt^2}=-Ea\left[\frac{\partial u}{\partial x}\right]_{x=l},$$

$$\text{with} \quad \xi=\frac{Sl}{E},\ \frac{d\xi}{dt}=0, \quad \text{when } t=0. \tag{2}$$

The subsidiary equation derived from (1) is

$$\frac{d^2\bar{u}}{dx^2}-\frac{p^2}{c^2}\bar{u}=-\frac{p}{c^2}\frac{Sx}{E}, \tag{3}$$

to be solved with

$$\bar{u}=0, \quad \text{when } x=0,$$

and, from (2),

$$mp^2\bar{u}=-Ea\frac{d\bar{u}}{dx}+\frac{mpSl}{E}, \quad \text{when } x=l. \tag{4}$$

The solution of (3) which vanishes at $x=0$ is

$$\bar{u}=\frac{Sx}{pE}+A\sinh\frac{px}{c},$$

and, substituting in (4), we find the arbitrary constant

$$A=-\frac{Sla}{mcp^2\{(pl/c)\sinh(pl/c)+k\cosh(pl/c)\}},$$

where $k=lEa/mc^2=al\rho/m$, the ratio of the mass of the bar to the mass attached to its end. Hence

$$\bar{u}=\frac{Sx}{pE}-\frac{Sla\sinh(px/c)}{mcp^2[(pl/c)\sinh(pl/c)+k\cosh(pl/c)]}.$$

So, by the Inversion Theorem,

$$u=\frac{Sx}{E}-\frac{Sla}{2\pi imc}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}\sinh(\lambda x/c)\,d\lambda}{\lambda^2[(\lambda l/c)\sinh(\lambda l/c)+k\cosh(\lambda l/c)]}. \tag{5}$$

The integrand is a single-valued function of $\lambda$ and we use the contour† of Fig. 10. The poles of the integrand are $\lambda = 0$ (with residue $x/kc$) and $\lambda = \pm ic\alpha_s/l$, where $\alpha_s$, $s = 1, 2,...,$ are the roots (all real and simple) of $z \tan z = k$.

Now

$$\left[\frac{d}{d\lambda}\left(\frac{\lambda l}{c}\sinh\frac{\lambda l}{c}+k\cosh\frac{\lambda l}{c}\right)\right]_{\lambda=ic\alpha_s/l} = \frac{il}{c\alpha_s}(k+k^2+\alpha_s^2)\cos\alpha_s.$$

Therefore the residue at the pole $\lambda = ic\alpha_s/l$ is

$$-\frac{l}{c\alpha_s}e^{ic\alpha_s t/l}\frac{\sin x\alpha_s/l}{(k+k^2+\alpha_s^2)\cos\alpha_s}.$$

Thus finally, from (5),

$$u = \frac{2Sl^2a}{mc^2}\sum_{s=1}^{\infty}\frac{\cos\alpha_s ct/l \sin x\alpha_s/l}{\alpha_s(k+k^2+\alpha_s^2)\cos\alpha_s}.$$

**63.** *Bar of length $l$ with a mass $m$ attached to the end $x = l$. At $t = 0$, when the bar is moving with velocity $U$ in the direction of its length, the end $x = 0$ is fixed. To find the tension in the bar at the origin.*‡

Let $u$ be the displacement of the point $x$ of the bar, and $\xi$ that of the mass $m$, so that $\xi = \lim_{x\to l} u$.

The equation of motion of the bar is

$$\left.\begin{aligned}&\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < l,\ t > 0,\\ \text{with}\quad &u = 0,\ \frac{\partial u}{\partial t} = U, \quad \text{when } t = 0,\ 0 < x < l,\end{aligned}\right\} \tag{1}$$

and $\quad u = 0, \quad$ when $x = 0,\ t > 0$.

If $a$ is the area of the bar, the equation of motion of the mass $m$ is

$$\begin{aligned}&m\frac{d^2\xi}{dt^2} = -Ea\left[\frac{\partial u}{\partial x}\right]_{x=l},\\ \text{with}\quad &\xi = 0,\ \frac{d\xi}{dt} = U, \quad \text{when } t = 0.\end{aligned} \tag{2}$$

† For the solution of a problem of this type in terms of waves see § 63.
‡ Perry, 'Winding Ropes in Mines', *Phil. Mag.* (6), **11** (1906), 107.

The subsidiary equation corresponding to (1) is

$$\frac{d^2\bar{u}}{dx^2}-\frac{p^2}{c^2}\bar{u} = -\frac{U}{c^2}, \tag{3}$$

with $\bar{u} = 0$, when $x = 0$,

and, from (2),

$$mp^2\bar{u} = -Ea\frac{d\bar{u}}{dx}+mU, \quad \text{when } x = l. \tag{4}$$

The solution of (3) which vanishes at $x = 0$ is

$$\bar{u} = \frac{U}{p^2}(1-\cosh qx)+A\sinh qx, \tag{5}$$

where $q = p/c$.

Substituting in (4) we find

$$A = \frac{U}{p^2}\frac{ql\cosh ql+k\sinh ql}{ql\sinh ql+k\cosh ql},$$

where $k = Eal/mc^2 = al\rho/m$.

Thus, from (5),

$$\bar{u} = \frac{2U}{p^2}\sinh\tfrac{1}{2}qx\frac{ql\cosh q(l-\frac{1}{2}x)+k\sinh q(l-\frac{1}{2}x)}{ql\sinh ql+k\cosh ql}. \tag{6}$$

The tension at any point of the bar is given by $X = E(\partial u/\partial x)$, so at $x = 0$ it is

$$X_0 = E\left[\frac{\partial u}{\partial x}\right]_{x=0},$$

and thus

$$\begin{aligned}\bar{X}_0 &= E\left[\frac{d\bar{u}}{dx}\right]_{x=0}\\ &= \frac{EU}{cp}\frac{ql\cosh ql+k\sinh ql}{ql\sinh ql+k\cosh ql}\\ &= \frac{EU}{cp}\left[1+2\frac{p-b}{p+b}e^{-2pl/c}+2\left(\frac{p-b}{p+b}\right)^2e^{-4pl/c}+\ldots\right],\end{aligned} \tag{7}$$

where $b$ is written for $ck/l$, and we have expanded in a series of exponentials in order to obtain a solution in terms of multiply reflected waves as in § 44.

We apply § 3, Theorem V, to the terms of (7) successively, using for shortness the notation

$$\left.\begin{aligned}H(t) &= 0, \quad \text{when } t < 0,\\ H(t) &= 1, \quad \text{when } t > 0.\end{aligned}\right\} \tag{8}$$

Thus we have

$$X_0 = \frac{EU}{c}+\frac{2EU}{c}[2e^{-b(t-2l/c)}-1]H(t-2l/c)+$$
$$+\frac{2EU}{c}\left[1-4b\left(t-\frac{4l}{c}\right)e^{-b(t-4l/c)}\right]H(t-4l/c)+\ldots.$$

The second term is zero for $0 < t < 2l/c$, i.e. until the wave reflected from $x = l$ arrives; the third term is zero for $0 < t < 4l/c$, i.e. until the three-times-reflected wave arrives, and so on.

The displacement at any point may be found in the same way; we calculate it for $x = l$. Putting $x = l$ in (6) we have

$$[\bar{u}]_{x=l} = \frac{U}{p^2}-\frac{kU}{p^2}\,\frac{1}{ql\sinh ql+k\cosh ql}$$
$$= \frac{U}{p^2}-\frac{2ckU}{lp^2(p+b)}e^{-pl/c}\left[1+\left(\frac{p-b}{p+b}\right)e^{-2pl/c}+\left(\frac{p-b}{p+b}\right)^2 e^{-4pl/c}+\ldots\right],$$

where $b = kc/l$. Therefore

$$u_{x=l} = Ut-\frac{2lU}{kc}\left[b\left(t-\frac{l}{c}\right)-1+e^{-b(t-l/c)}\right]H\left(t-\frac{l}{c}\right)$$
$$-\frac{2lU}{kc}\left[3\left(t-\frac{3l}{c}\right)-3e^{-b(t-3l/c)}-2b\left(t-\frac{3l}{c}\right)e^{-b(t-3l/c)}\right]H\left(t-\frac{3l}{c}\right)-$$

**64.** *Two equal rods*† *of length $l$, moving longitudinally in opposite directions with equal speeds, collide. To find the subsequent motion.*

Suppose the collision takes place at $t = 0$, at the origin. Then by symmetry it is sufficient to consider one of the rods: we choose the rod $0 < x < l$ and suppose its initial velocity to be $-U$.

So long as the rods are in contact we have to solve

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < l,\ t > 0,$$

† The problem of a bar struck at one end by a particle moving in the direction of the length of the bar is considered by Bromwich, *Proc. Lond. Math. Soc.* (2), **15** (1914), 427.

with
$$u = 0, \quad \text{at } x = 0,\, t > 0,$$
$$\frac{\partial u}{\partial x} = 0, \quad \text{at } x = l,\, t > 0,$$
$$u = 0,\, \frac{\partial u}{\partial t} = -U, \quad \text{when } t = 0,\, 0 < x < l.$$

The subsidiary equation is
$$\frac{d^2\bar{u}}{dx^2} - \frac{p^2}{\;}\bar{u} = \frac{U}{\;}$$
with
$$\frac{d\bar{u}}{dx} = 0,\, x = l, \quad \text{and} \quad \bar{u} = 0,\, x = 0.$$

The solution is
$$\bar{u} = -\frac{U}{p^2}\left\{1 - \frac{\cosh p(l-x)/c}{\cosh pl/c}\right\}$$
$$= -\frac{U}{p^2} + \frac{U}{p^2}\{e^{-px/c} + e^{-p(2l-x)/c} - e^{-p(2l+x)/c} - e^{-p(4l-x)/c} + \ldots\}. \tag{1}$$

Therefore
$$u = -Ut + U\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right) + U\left(t - \frac{2l-x}{c}\right)H\left(t - \frac{2l-x}{c}\right) -$$
$$- U\left(t - \frac{2l+x}{c}\right)H\left(t - \frac{2l+x}{c}\right) - U\left(t - \frac{4l-x}{c}\right)H\left(t - \frac{4l-x}{c}\right) + \ldots, \tag{2}$$
where $H(t)$ is defined in § 63 (8).

This solution is only valid while the rods are in contact, i.e. while the pressure between them is positive, or $[\partial u/\partial x]_{x=0} < 0$.

To find when contact ceases, we have from (1)
$$\left[\frac{d\bar{u}}{dx}\right]_{x=0} = -\frac{U}{cp}\tanh pl/c$$
$$= -\frac{U}{cp}\{1 - 2e^{-2pl/c} + 2e^{-4pl/c} - \ldots\}.$$

So
$$\left[\frac{\partial u}{\partial x}\right]_{x=0} = -\frac{U}{c} + \frac{2U}{c}H\left(t - \frac{2l}{c}\right) - \ldots.$$

Therefore $[\partial u/\partial x]_{x=0} < 0$ for $0 < t < 2l/c$, and at $t = 2l/c$ it becomes positive and the rods separate. Thus the solution (2)

holds only for $0 < t < 2l/c$, and so only the first three terms are needed and we have

$$u = -Ut + U\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right) + U\left(t-\frac{2l-x}{c}\right)H\left(t-\frac{2l-x}{c}\right),$$

when $0 < t < 2l/c$.

Similarly,

$$\frac{\partial u}{\partial t} = -U + UH\left(t-\frac{x}{c}\right) + UH\left(t-\frac{2l-x}{c}\right), \quad \text{when } 0 < t < \frac{2l}{c}.$$

From these it follows that, when $t \to 2l/c$,

$$u \to 0, \quad 0 < x < l,$$

$$\frac{\partial u}{\partial t} \to U, \quad 0 < x < l,$$

i.e. the rod is unstrained and moving with velocity $U$. These are the initial conditions for the subsequent motion. Clearly the rod rebounds without vibration and with velocity $U$.

**65.** *Transverse vibration of beams.*

The approximate differential equation for transverse vibration of a uniform beam is

$$\frac{\partial^4 u}{\partial x^4} + \frac{1}{k^2}\frac{\partial^2 u}{\partial t^2} = \frac{P}{EI}, \tag{1}$$

where $u$ is the displacement at the point $x$ of the beam, $P(x,t)$ the externally applied force (including gravity if this is not neglected), $\rho$ and $E$ the density and Young's modulus of the material, $S$ and $I$ the area and moment of inertia of the cross-section of the beam, and $k^2 = EI/(\rho S)$.

We consider throughout the case of a beam freely hinged at its ends, $x = 0$ and $x = l$, in which case the boundary conditions are

$$u = \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{when } x = 0, \tag{2}$$

and

$$u = \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{when } x = l. \tag{3}$$

Then, if the initial displacement and velocity of the point $x$ of the beam are $f(x)$ and $g(x)$ respectively, the subsidiary equation is

$$\frac{d^4\bar{u}}{dx^4}+\frac{p^2}{k^2}\bar{u} = \frac{\bar{P}}{EI}+\frac{1}{k^2}[pf(x)+g(x)], \tag{4}$$

with

$$\bar{u} = \frac{d^2\bar{u}}{dx^2} = 0, \quad \text{when } x = 0, \tag{5}$$

and

$$u = \frac{d^2\bar{u}}{dx^2} = 0, \quad \text{when } x = l. \tag{6}$$

Writing†

$$q^4 = -p^2/k^2, \tag{7}$$

and

$$\frac{\bar{P}}{EI}+\frac{1}{k^2}[pf(x)+g(x)] = \phi(x), \tag{8}$$

we have to solve‡

$$\frac{d^4\bar{u}}{dx^4}-q^4\bar{u} = \phi(x) \tag{9}$$

with the boundary conditions (5) and (6).

Now

$$\frac{1}{p^4-q^4} = \frac{1}{2q^2}\left\{\frac{1}{p^2-q^2}-\frac{1}{p^2+q^2}\right\} = \frac{1}{2q^3}\int\limits_0^\infty e^{-px}[\sinh qx-\sin qx]\,dx;$$

thus, using § 3, Theorem VI, a Particular Integral of (9) is

$$\frac{1}{2q^3}\int\limits_0^x \phi(\xi)[\sinh q(x-\xi)-\sin q(x-\xi)]\,d\xi.$$

Adding the complementary function, the general solution of (9) is

$$\bar{u} = A\sinh qx+B\cosh qx+C\sin qx+D\cos qx+$$
$$+\frac{1}{2q^3}\int\limits_0^x \phi(\xi)[\sinh q(x-\xi)-\sin q(x-\xi)]\,d\xi. \tag{10}$$

† The minus sign in (7) is introduced since the differential equation (9) is a little easier to handle than $(D^4+a^4)y = \phi(x)$.

‡ As an alternative to this method, Variation of Parameters as employed in § 42 may be used.

The conditions (5) require $B = D = 0$, and then by (6) we must have

$$A\sinh ql + C\sin ql + \frac{1}{2q^3}\int^{l} \phi(\xi)[\sinh q(l-\xi) - \sin q(l-\xi)]\, d\xi = 0,$$

and

$$A\sinh ql - C\sin ql + \frac{1}{2q^3}\int_0^{l} \phi(\xi)[\sinh q(l-\xi) + \sin q(l-\xi)]\, d\xi = 0.$$

Solving for $A$ and $C$ and substituting in (10), we have finally

$$\bar{u} = \frac{1}{2q^3}\int_0^{l} \phi(\xi)\left\{\frac{\sin q(l-\xi)\sin qx}{\sin ql} - \frac{\sinh q(l-\xi)\sinh qx}{\sinh ql}\right\} d\xi +$$

$$+\frac{1}{2q^3}\int^{x} \phi(\xi)\{\sinh q(x-\xi) - \sin q(x-\xi)\}\, d\xi$$

$$-\frac{}{2q^3\sin ql\sinh ql}\times$$

$$\times\int^{x} \phi(\xi)[\sin q(l-x)\sin q\xi\sinh ql - \sinh q(l-x)\sinh q\xi\sin ql]\, d\xi +$$

$$+\frac{}{2q^3\sin ql\sinh ql}\times$$

$$\times\int_x^{l} \phi(\xi)[\sin q(l-\xi)\sin qx\sinh ql - \sinh q(l-\xi)\sinh qx\sin ql]\, d\xi. \tag{11}$$

In the following examples we shall determine $u$ from this for particular values of $\phi(x)$. The general case may be treated in the same way.

Ex. 1. *The beam under no external forces and initially straight. At $t = 0$ velocity $v$ is given to a small length $\epsilon$ at $x'$ (e.g. by an impulse).*†

Here, from (8), $\phi(x)$ equals $v/k^2$ in the small length $\epsilon$ at $x'$ and is zero outside this. So in (11), if $0 < x < x'$, the first

† Timoshenko, loc. cit., p. 230.

integral vanishes and in the second the integrand may be given its value at $\xi = x'$ since $\epsilon$ is small. Thus we obtain

$$\bar{u} = \frac{v\epsilon}{2k^2}\frac{\sin q(l-x')\sin qx \sinh ql - \sinh q(l-x')\sinh qx \sin ql}{q^3 \sin ql \sinh ql},$$
$$0 < x < x',$$

where, by (7), $q = \sqrt{(ip/k)}$, and, if $x' < x < l$, we have to interchange $x$ and $x'$.

Thus, using the Inversion Theorem, we have

$$u = \frac{v\epsilon}{4\pi i k^2}\times$$
$$\times\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\sin \mu(l-x')\sin \mu x \sinh \mu l - \sinh \mu(l-x')\sinh \mu x \sin \mu l}{\mu^3 \sin \mu l \sinh \mu l}\,d\lambda, \tag{12}$$

where
$$\mu = \sqrt{\left(\frac{i\lambda}{k}\right)}. \tag{13}$$

It follows from the series for $\sin z$ and $\sinh z$ that the integrand of (12) is a single-valued function of $\lambda$ with no pole at $\lambda = 0$. To find its poles we notice that the region $\pi > \arg\lambda > -\pi$ of the $\lambda$-plane corresponds to $\frac{3}{4}\pi > \arg\mu > -\frac{1}{4}\pi$ of the $\mu$-plane. Thus the required poles are at the zeros of $\sin \mu l \sinh \mu l$ in this region, namely,

$$\mu = \frac{n\pi}{l}, \quad \text{i.e. } \lambda = \frac{kn^2\pi^2}{il^2}, \quad n = 1, 2, 3,..., \tag{14}$$

and
$$\mu = \frac{in\pi}{l}, \quad \text{i.e. } \lambda = -\frac{kn^2\pi^2}{il^2}, \quad n = 1, 2, 3,.... \tag{15}$$

The residue at the pole $\lambda = kn^2\pi^2/(il^2)$ is

$$\frac{2ikl}{n^2\pi^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi x'}{l}e^{-ikn^2\pi^2t/l^2}, \tag{16}$$

and that at the pole $\lambda = -kn^2\pi^2/(il^2)$ is

$$-\frac{2ikl}{n^2\pi^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi x'}{l}e^{ikn^2\pi^2t/l^2}. \tag{17}$$

Therefore, using the contour of Fig. 10, we obtain in the usual way

$$u = \frac{2lv\epsilon}{\pi^2 k}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi x'}{l}\sin\frac{kn^2\pi^2 t}{l^2}, \quad 0 < x < x', \tag{18}$$

and this result, being symmetrical in $x$ and $x'$, holds also for $x' < x < l$.

Ex. 2. *The beam initially straight and at rest. To find the vibration due to suddenly applying a load $W$ at $x'$ (gravity forces on the beam being neglected).*

Suppose the loading is $W/\epsilon$ per unit length over a small length $\epsilon$ about $x'$, and zero elsewhere. Then, by (8), $\phi(x) = W/(\epsilon EIp)$ in this region and is zero outside it. Thus, from (11),

$$\bar{u} = \frac{W}{2EI} \cdot \frac{\sin q(l-x')\sin qx \sinh ql - \sinh q(l-x')\sinh qx \sin ql}{pq^3 \sin ql \sinh ql}$$

$$0 < x < x'. \quad (19)$$

Therefore, by the Inversion Theorem,

$$u = \frac{W}{4\pi i EI} \times$$

$$\times \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{\sin \mu(l-x')\sin \mu x \sinh \mu l - \sinh \mu(l-x')\sinh \mu x \sin \mu l}{\lambda \mu^3 \sin \mu l \sinh \mu l}\, d\lambda, \quad (20)$$

where $\mu = \sqrt{(i\lambda/k)}$.

The integrand of (20) has a pole at $\lambda = 0$ in addition to those given by (14) and (15).

The residue at the pole $\lambda = 0$ is

$$\frac{1}{3l} x(l-x')(2lx'-x^2-x'^2),$$

and those at the poles $\lambda = \pm kn^2\pi^2/(il^2)$ are obtained by multiplying (16) and (17) by $\pm il^2/(kn^2\pi^2)$ respectively.

Thus, finally, if $0 < x < x'$,

$$u = \frac{W}{6EIl} x(l-x')(2lx'-x^2-x'^2) -$$

$$- \frac{2Wl^3}{EI\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin \frac{n\pi x'}{l} \sin \frac{n\pi x}{l} \cos \frac{kn^2\pi^2 t}{l^2}, \quad (21)$$

and, if $x' < x < l$, we have to interchange $x$ and $x'$ in this result. The first term of (21) is, of course, the static deflexion.

Ex. 3. *The vibration due to a pulsating force $P_0 \sin \omega t$ applied at $t = 0$ at $x'$, the beam being initially straight and at rest.*†

† Timoshenko, loc. cit., p. 239.

The solution for this case may be obtained from (19) by replacing $W/p$ by $\omega P_0/(p^2+\omega^2)$ so that we get in place of (20)

$$u = \frac{\omega P_0}{4\pi i EI}\times$$
$$\times\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\sin\mu(l-x')\sin\mu x\sinh\mu l-\sinh\mu(l-x')\sinh\mu x\sin\mu l}{\mu^3(\lambda^2+\omega^2)\sin\mu l\sinh\mu l}\,d\lambda, \tag{22}$$

where $0 < x < x'$, and $\mu = \sqrt{(i\lambda/k)}$.

The integrand of (22) has poles at $\lambda = \pm i\omega$ as well as those given by (14) and (15).

The residue at the pole $\lambda = i\omega$ is

$$-e^{i\omega t}\frac{\sinh(l-x')\sqrt{(\omega/k)}\sinh x\sqrt{(\omega/k)}\sin l\sqrt{(\omega/k)}-\sin(l-x')\sqrt{(\omega/k)}\sin x\sqrt{(\omega/k)}\sinh l\sqrt{(\omega/k)}}{2i\omega(\omega/k)^{\frac{3}{2}}\sin l\sqrt{(\omega/k)}\sinh l\sqrt{(\omega/k)}}.$$

Also, the residues at the poles $\lambda = \pm kn^2\pi^2/(il^2)$ are obtained by multiplying (16) and (17) by†

$$\frac{1}{\omega^2-k^2n^4\pi^4/l^4} = -\frac{l^4}{k^2n^4\pi^4-\omega^2l^4}.$$

So, finally,

$$u = \frac{P_0 k^{\frac{3}{2}}}{2EI\omega^{\frac{3}{2}}}\frac{\sin\omega t}{\sin l\sqrt{(\omega/k)}\sinh l\sqrt{(\omega/k)}}\times$$
$$\times\{\sin(l-x')\sqrt{(\omega/k)}\sin x\sqrt{(\omega/k)}\sinh l\sqrt{(\omega/k)}-$$
$$-\sinh(l-x')\sqrt{(\omega/k)}\sinh x\sqrt{(\omega/k)}\sin l\sqrt{(\omega/k)}\}-$$
$$-\frac{2k\omega l^5P_0}{\pi^2 EI}\sum_{n=1}^{\infty}\frac{1}{n^2(k^2n^4\pi^4-\omega^2l^4)}\sin\frac{n\pi x'}{l}\sin\frac{n\pi x}{l}\sin\frac{kn^2\pi^2t}{l^2},$$
$$0 < x < x', \tag{23}$$

and for $x' < x < l$ we interchange $x$ and $x'$ in (23).

Ex. 4. *The beam initially straight and at rest. At $t = 0$ a concentrated load $W$ starts from $x = 0$ and moves along the beam with uniform velocity $v$.*‡

† It is assumed that $\omega^2 \neq k^2n^4\pi^4/l^4$ for any integral $n$, i.e. that the applied force is not in resonance with any natural frequency of the beam. If this is not the case, the integrand of (22) has double poles at $\lambda = \pm i\omega$ and a separate calculation must be made.

‡ Timoshenko, loc. cit., p. 242.

We regard the load $W$ as a uniform loading $W/\epsilon$ per unit length spread over a small length $\epsilon$.

Then

$$P(x,t) = W/\epsilon \quad \text{for} \quad \frac{1}{v}(x-\tfrac{1}{2}\epsilon) < t < \frac{1}{v}(x+\tfrac{1}{2}\epsilon),$$

and is zero at all other times.

Thus

$$\bar{P}(x) = \frac{W}{\epsilon}\int\limits_{(x-\frac{1}{2}\epsilon)/v}^{(x+\frac{1}{2}\epsilon)/v} e^{-pt}\,dt$$

$$= \frac{W}{v}e^{-px/v},$$

neglecting the square of the small quantity $\epsilon$.

So (11) becomes

$$\bar{u} = \frac{W}{2EIv}\int\limits_0^x e^{-p\xi/v}\times$$

$$\times\frac{\sin q(l-x)\sin q\xi\sinh ql-\sinh q(l-x)\sinh q\xi\sin ql}{q^3\sin ql\sinh ql}\,d\xi +$$

$$+\frac{W}{2EIv}\int\limits_x^l e^{-p\xi/v}\times$$

$$\times\frac{\sin q(l-\xi)\sin qx\sinh ql-\sinh q(l-\xi)\sinh qx\sin ql}{q^3\sin ql\sinh ql}\,d\xi.$$

Therefore, applying the Inversion Theorem and interchanging the orders of integration, we have

$$u = \frac{W}{2EIv}\int\limits_0^x \frac{d\xi}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda(t-\xi/v)}\times$$

$$\times\frac{\sin\mu(l-x)\sin\mu\xi\sinh\mu l-\sinh\mu(l-x)\sinh\mu\xi\sin\mu l}{\mu^3\sin\mu l\sinh\mu l}\,d\lambda +$$

$$+\frac{W}{2EIv}\int\limits_x^l \frac{d\xi}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda(t-\xi/v)}\times$$

$$\times\frac{\sin\mu(l-\xi)\sin\mu x\sinh\mu l-\sinh\mu(l-\xi)\sinh\mu x\sin\mu l}{\mu^3\sin\mu l\sinh\mu l}\,d\lambda, \quad (24)$$

where $\mu = \sqrt{(i\lambda/k)}$.

Now since the integrand of the second line integral of (24) differs from that of (12) only by the factor $e^{-\lambda\xi/v}$, we have from (18) and § 3, Theorem V,

$$\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda(t-\xi/v)}\times$$
$$\times\frac{\sin\mu(l-\xi)\sin\mu x\sinh\mu l-\sinh\mu(l-\xi)\sinh\mu x\sin\mu l}{\mu^3\sin\mu l\sinh\mu l}\,d\lambda$$
$$= 0, \quad \text{when } t < \xi/v,$$
$$= \frac{4kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\sin\frac{kn^2\pi^2}{l^2}\left(t-\frac{\xi}{v}\right), \quad \text{when } t > \xi/v,$$

and this result holds also for the first line integral in (24) since (18) is symmetrical in $x$ and $x'$.

Thus (24) becomes

$$= \frac{2klW}{\pi^2EIv}\int\limits_0 d\xi\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi\xi}{l}\sin\frac{kn^2\pi^2}{l^2}\left(t-\frac{\xi}{v}\right)$$
$$= \frac{2klW}{\pi^2EIv}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{l}\int\limits_0^{vt}\sin\frac{n\pi\xi}{l}\sin\frac{kn^2\pi^2}{l^2}\left(t-\frac{\xi}{v}\right)d\xi$$
$$-\frac{2k^2l^3W}{\pi^2EI}\sum_{n=1}^{\infty}\frac{1}{n^2(l^2v^2-k^2n^2\pi^2)}\sin\frac{n\pi x}{l}\sin\frac{n\pi vt}{l}+$$
$$+\frac{2kvl^4W}{\pi^3EI}\sum_{n=1}^{\infty}\frac{1}{n^3(l^2v^2-k^2n^2\pi^2)}\sin\frac{n\pi x}{l}\sin\frac{kn^2\pi^2t}{l^2},$$

provided $v \neq kn\pi/l$ for any $n$. If $v = km\pi/l$ for some integer $m$, the terms $n = m$ of the series are to be omitted and a term

$$\frac{Wl^3}{EI\pi^4m^4}\left\{\sin\frac{m\pi vt}{l}-\frac{m\pi vt}{l}\cos\frac{m\pi vt}{l}\right\}\sin\frac{m\pi x}{l}$$

added.

**66.** *Semi-infinite beam $x > 0$ initially straight and at rest. At $t = 0$ the end $x = 0$ is given a small displacement $a$.*

We have to solve

$$\frac{\partial^4 u}{\partial x^4}+\frac{1}{k^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad x > 0, t > 0,$$

with $$u = a, \quad \text{when } x = 0,\ t > 0,$$
$$\frac{\partial^2 u}{\partial x^2} = 0, \quad \text{when } x = 0,\ t > 0.$$

The subsidiary equation is
$$\frac{d^4\bar{u}}{dx^4}+\frac{p^2}{k^2}\bar{u} = 0, \quad x > 0, \tag{1}$$
with $$\bar{u} = \frac{a}{p}, \qquad \frac{d^2\bar{u}}{dx^2} = 0, \quad \text{when } x = 0. \tag{2}$$

The solution of (1) which remains finite as $x \to \infty$ is
$$\bar{u} = Ae^{-qx}\sin qx+Be^{-qx}\cos qx,$$
where $q = \sqrt{(p/2k)}$.

The conditions (2) require $B = a/p$ and $A = 0$. Thus
$$\bar{u} = \frac{a}{p}e^{-qx}\cos qx.$$
Therefore, using the Inversion Theorem,
$$u = \frac{a}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t-x\sqrt{(\lambda/2k)}}\cos x\sqrt{(\lambda/2k)}\,\frac{d\lambda}{\lambda}. \tag{3}$$

The integrand of (3) has a branch-point at $\lambda = 0$, so we use the contour of Fig. 11. There are no poles within this path, and it may be verified that the integrals over $AC$ and $BF$ vanish when $R \to \infty$. So the integral in (3) is the sum of integrals along $CD$, the small circle, and $EF$.

From $CD$ and $EF$ we obtain, putting $\lambda = 2ku^2e^{-i\pi}$ on $CD$, and $\lambda = 2ku^2e^{i\pi}$ on $EF$,
$$\frac{a}{i\pi}\int\limits_{\infty}^{0} e^{-2ku^2t+ixu}\cosh xu\frac{du}{u}+\frac{a}{i\pi}\int\limits_{0}^{\infty} e^{-2ku^2t-ixu}\cosh xu\,\frac{du}{u}$$
$$= -\frac{2a}{\pi}\int\limits_{0}^{\infty} e^{-2ku^2t}\sin xu\cosh xu\,\frac{du}{u}$$
$$= -\frac{a}{\sqrt{\pi}}\int\limits_{0}^{x(2kt)^{-\frac{1}{2}}} \{\cos\tfrac{1}{2}y^2+\sin\tfrac{1}{2}y^2\}\,dy.$$

The small circle gives $a$.

Thus finally

$$u = a - \frac{a}{\sqrt{\pi}} \int\limits^{x(2kt)^{-\frac{1}{2}}} \{\cos \tfrac{1}{2}y^2 + \sin \tfrac{1}{2}y^2\}\, dy.$$

**67.** *A doubly infinite string stretched along the x-axis has a particle of mass m attached to it at the origin; transverse motion of the particle is resisted by a force $m\mu^2$ times the displacement. At $t = 0$, when the system is at rest in the equilibrium position, the particle is given a transverse velocity U; to find the subsequent motion.*†

Let $y$ be the displacement of the point $x$ of the string and $\xi$ that of the particle, so that $\xi = \lim_{x\to 0} y$. We need consider only $x > 0$; the motion for $x < 0$ will be symmetrical. Let $T$ be the tension of the string, $\rho$ its density, and $c^2 = T/\rho$. Then the equation of motion of the string is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0, \quad x > 0,\, t > 0,$$

with $$y = \frac{\partial y}{\partial t} = 0, \quad \text{when } t = 0.$$

The equation of motion of the particle is

$$m\frac{d^2\xi}{dt^2} + m\mu^2\xi = 2T\left[\frac{\partial y}{\partial x}\right]_{x=0},$$

with $$\xi = 0, \quad \frac{d\xi}{dt} = U, \qquad \text{when } t = 0.$$

The subsidiary equation is

$$\frac{d^2\bar{y}}{dx^2} - \frac{p^2}{c^2}\bar{y} = 0, \quad x > 0, \tag{1}$$

with $$m(p^2+\mu^2)\bar{y} - 2T\frac{d\bar{y}}{dx} = mU, \quad \text{when } x = 0. \tag{2}$$

The solution of (1) which is finite as $x \to \infty$ is

$$\bar{y} = Ae^{-px/c}.$$

† Lamb, *Proc. Lond. Math. Soc.* (1), **32** (1900), p. 208.

Substituting this in (2) we find

$$A\left\{p^2+\mu^2+\frac{2T}{mc}p\right\} = U.$$

Thus writing $b = m/\rho$, so that $T/mc = c/b$, we have

$$\bar{y} = \frac{Ue^{-px/c}}{(p+c/b)^2+(\mu^2-c^2/b^2)}.$$

If $\mu^2 > c^2/b^2$, the solution is, by § 3, Theorems IV and V,

$$\left.\begin{aligned} y &= \frac{U}{\sqrt{(\mu^2-c^2/b^2)}}e^{-(ct-x)/b}\sin(t-x/c)\sqrt{(\mu^2-c^2/b^2)}, \quad x < ct, \\ &= 0, \quad x > ct. \end{aligned}\right\}$$

If $\mu^2 < c^2/b^2$, the solution is

$$\left.\begin{aligned} y &= \frac{U}{\sqrt{(c^2/b^2-\mu^2)}}e^{-(ct-x)/b}\sinh(t-x/c)\sqrt{(c^2/b^2-\mu^2)}, \quad x < ct, \\ &= 0, \quad x > ct. \end{aligned}\right\}$$

And if $\mu^2 = c^2/b^2$,

$$\left.\begin{aligned} y &= U(t-x/c)e^{-\mu(t-x/c)}, \quad x < ct, \\ &= 0, \quad x > ct. \end{aligned}\right\}$$

**68.** *A heavy chain is hung from one end. To find the motion from given initial conditions.*

Let the origin be the equilibrium position of the free end, $x = l$ the fixed end, and $y$ the horizontal displacement of the point $x$ of the chain. Then the equation of motion is†

$$\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right)-\frac{1}{g}\frac{\partial^2 y}{\partial t^2} = 0.$$

Suppose that $y = f(x)$, $\partial y/\partial t = 0$, when $t = 0$. Then the subsidiary equation is

$$\frac{d}{dx}\left(x\frac{d\bar{y}}{dx}\right)-\frac{p^2}{g}\bar{y} = -\frac{p}{g}f(x), \tag{1}$$

with

$$\bar{y} = 0, \quad \text{when } x = l, \quad \text{and} \quad \bar{y} \text{ finite}, \quad \text{when } x = 0. \tag{2}$$

† Lamb, *Higher Mechanics* (2nd ed., 1929), p. 225.

To solve this we seek the Green's function† $G(x, \xi)$ which is to satisfy the homogeneous equation corresponding to (1), namely,

$$\frac{d}{dx}\left(x\frac{dG}{dx}\right)-\frac{p^2}{g}G = 0, \tag{3}$$

except at the point $x = \xi$, where we are to have

$$[G]_{\xi+0}-[G]_{\xi-0} = 0 \tag{4}$$

and

$$\left[\frac{\partial G}{\partial x}\right]_{\xi+0}-\left[\frac{\partial G}{\partial x}\right]_{\xi-0} = \frac{1}{\xi}. \tag{5}$$

Also, at $x = 0$ and $x = l$, $G$ is to satisfy the same boundary conditions (2) as $\bar{y}$, i.e.

$$G(x, \xi) = 0, \quad \text{when } x = l \tag{6}$$

and

$$G(x, \xi) \text{ is to be finite,} \quad \text{when } x = 0. \tag{7}$$

To solve (3), put $z = 2p(x/g)^{\frac{1}{2}}$ and we have

$$\frac{d^2G}{dz^2}+\frac{1}{z}\frac{dG}{dz}-G = 0,$$

solutions of which are $I_0(z)$ and $K_0(z)$.

Thus we assume

$$\begin{aligned} G(x, \xi) &= AI_0\left(2p\sqrt{\frac{x}{g}}\right), \quad 0 \leqslant x \leqslant \xi, \\ &= BI_0\left(2p\sqrt{\frac{x}{g}}\right)+CK_0\left(2p\sqrt{\frac{x}{g}}\right), \quad \xi \leqslant x \leqslant l, \end{aligned} \tag{8}$$

which satisfies the condition (7). The other three conditions (6), (4), and (5) require

$$BI_0\left(2p\sqrt{\frac{l}{g}}\right)+CK_0\left(2p\sqrt{\frac{l}{g}}\right) = 0,$$

$$AI_0\left(2p\sqrt{\frac{\xi}{g}}\right)-BI_0\left(2p\sqrt{\frac{\xi}{g}}\right)-CK_0\left(2p\sqrt{\frac{\xi}{g}}\right) = 0,$$

$$-AI_0'\left(2p\sqrt{\frac{\xi}{g}}\right)+BI_0'\left(2p\sqrt{\frac{\xi}{g}}\right)+CK_0'\left(2p\sqrt{\frac{\xi}{g}}\right) = \frac{\sqrt{g}}{p\sqrt{\xi}}.$$

Solving for $A$, $B$, $C$, using the result

$$I_0(z)K_0'(z)-K_0(z)I_0'(z) = -\frac{1}{z}, \tag{9}$$

† Alternatively 'Variation of Parameters' may be used; for this see § 42.

and substituting in (8), we obtain

$$G(x,\xi)$$
$$= -2\frac{I_0\{2p\sqrt{(x/g)}\}}{I_0\{2p\sqrt{(l/g)}\}}\left\{I_0\left(2p\sqrt{\frac{l}{g}}\right)K_0\left(2p\sqrt{\frac{\xi}{g}}\right)-K_0\left(2p\sqrt{\frac{l}{g}}\right)I_0\left(2p\sqrt{\frac{\xi}{g}}\right)\right\},$$
$$0 \leqslant x \leqslant \xi,$$
$$= -2\frac{I_0\{2p\sqrt{(\xi/g)}\}}{I_0\{2p\sqrt{(l/g)}\}}\left\{I_0\left(2p\sqrt{\frac{l}{g}}\right)K_0\left(2p\sqrt{\frac{x}{g}}\right)-K_0\left(2p\sqrt{\frac{l}{g}}\right)I_0\left(2p\sqrt{\frac{x}{g}}\right)\right\},$$
$$\xi \leqslant x \leqslant l. \tag{10}$$

Having found $G(x,\xi)$, the solution of (1) and (2) is easily determined in terms of it. Multiply (1) by $G(x,\xi)$ and (3) by $\bar{y}$, subtract these equations and integrate with respect to $x$ from 0 to $l$, and we get

$$-\frac{p}{g}\int_0^l f(x)G(x,\xi)\,dx = \int_0^l \left\{G\frac{d}{dx}\left(x\frac{d\bar{y}}{dx}\right)-\bar{y}\frac{d}{dx}\left(x\frac{dG}{dx}\right)\right\}dx$$
$$= \left[xG\frac{d\bar{y}}{dx}-x\bar{y}\frac{dG}{dx}\right]_0^{\xi-0}+\left[xG\frac{d\bar{y}}{dx}-x\bar{y}\frac{dG}{dx}\right]_{\xi+0}^{l}$$
$$= \xi\bar{y}(\xi)\left[\frac{dG}{dx}\right]_{\xi-0}^{\xi+0}$$
$$= \bar{y}(\xi),$$

where, in the reduction, (2), (4), (5), (6) have been used.

Thus, writing $G(x,\xi;p)$ for $G(x,\xi)$ to emphasize its dependence on $p$, the solution of (1) and (2) is

$$\bar{y}(x,p) = -\frac{p}{g}\int_0^l G(\eta,x;p)f(\eta)\,d\eta.$$

Therefore, using the Inversion Theorem,

$$y = -\frac{1}{2i\pi g}\int_{\gamma-i\infty}^{\gamma+i\infty}\lambda e^{\lambda t}\,d\lambda\int_0^l G(\eta,x;\lambda)f(\eta)\,d\eta$$
$$= -\frac{1}{2i\pi g}\int_0^l f(\eta)\,d\eta\int_{\gamma-i\infty}^{\gamma+i\infty}\lambda e^{\lambda t}G(\eta,x;\lambda)\,d\lambda, \tag{11}$$

assuming that the orders of integration can be inverted.

Now the poles of $G(\eta, x; \lambda)$, regarded as a function of $\lambda$, are at the zeros of $I_0\{2\lambda\sqrt{(l/g)}\}$, i.e. $\lambda = \pm\frac{1}{2}i\alpha_s\sqrt{(g/l)}$, where $\pm\alpha_s$, $s = 1, 2, 3,\ldots$, are the zeros (all real and simple) of $J_0(z) = 0$.

If $0 \leqslant \eta \leqslant x$, using the first formula of (10), the residue of $\lambda e^{\lambda t}G(\eta, x; \lambda)$ at $\lambda = \frac{1}{2}i\alpha_s\sqrt{(g/l)}$ is

$$\frac{i\alpha_s g}{2l}e^{\frac{1}{2}i\alpha_s t\sqrt{(g/l)}}\frac{I_0\{i\alpha_s\sqrt{(\eta/l)}\}I_0\{i\alpha_s\sqrt{(x/l)}\}K_0(i\alpha_s)}{I_0(i\alpha_s)}$$

$$= -\frac{g}{2l}e^{\frac{1}{2}i\alpha_s t\sqrt{(g/l)}}\frac{J_0\{\alpha_s\sqrt{(\eta/l)}\}J_0\{\alpha_s\sqrt{(x/l)}\}}{J_1^2(\alpha_s)},$$

using (9) and the facts that $J_0'(z) = -J_1(z)$ and $I_0(iz) = J_0(z)$. The same result holds for $x \leqslant \eta \leqslant l$. Thus, using the path of Fig. 10, the line integral in (11) may be replaced by $2\pi i$ times the sum of the residues at these poles, i.e.

$$y = \frac{1}{l}\int_0^l f(\eta)\,d\eta\sum_{s=1}^{\infty}\cos\{\tfrac{1}{2}\alpha_s t\sqrt{(g/l)}\}\frac{J_0\{\alpha_s\sqrt{(x/l)}\}J_0\{\alpha_s\sqrt{(\eta/l)}\}}{J_1^2(\alpha_s)}$$

$$= \frac{1}{l}\sum_{s=1}^{\infty}\cos\{\tfrac{1}{2}\alpha_s t\sqrt{(g/l)}\}\frac{J_0\{\alpha_s\sqrt{(x/l)}\}}{J_1^2(\alpha_s)}\int_0^l f(\eta)J_0\{\alpha_s\sqrt{(\eta/l)}\}\,d\eta,$$

provided $f(x)$ is such that the orders of integration and summation may be interchanged.

**69.** *A circular membrane of radius $a$ is stretched by tension $T$ and at rest in its equilibrium position. At $t = 0$ a uniform pressure $P_0\sin t$ is applied to the surface. It is required to find the motion.*

Let $T$ be the tension and $\rho$ the surface density of the membrane, $c^2 = T/\rho$. Then, if $u$ is the displacement of the membrane at radius $r$, the equation of motion is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\frac{P_0\sin\omega t}{T}, \quad 0 \leqslant r < a,\ t > 0,$$

with $\qquad u = 0, \quad r = a, \quad t \geqslant 0,$

and $\qquad u = \dfrac{\partial u}{\partial t} = 0, \quad \text{when } t = 0,\ 0 \leqslant r < a.$

The subsidiary equation is

$$\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}-\frac{p^2}{c^2}\bar{u} = -\frac{\quad}{T(p^2+\omega^2)}, \tag{1}$$

to be solved with $\bar{u} = 0$, when $r = a$.

The solution of (1) finite at the origin is

$$\bar{u} = \frac{P_0\,\omega c^2}{Tp^2(p^2+\omega^2)}+AI_0(pr/c),$$

and the condition $\bar{u} = 0$, when $r = a$, requires

$$A = -\frac{P_0\,\omega c^2}{Tp^2(p^2+\omega^2)}\frac{1}{I_0(pa/c)}.$$

Thus
$$\bar{u} = \frac{P_0\,\omega c^2}{Tp^2(p^2+\omega^2)}\left\{1-\frac{I_0(pr/c)}{I_0(pa/c)}\right\},$$

and therefore, using the Inversion Theorem for the second term,

$$u = \frac{P_0\,c^2}{\omega T}\left(t-\frac{1}{\omega}\sin\omega t\right)-\frac{P_0\,\omega c^2}{2\pi i T}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}I_0(\lambda r/c)\,d\lambda}{\lambda^2(\lambda^2+\omega^2)I_0(\lambda a/c)}.$$

The integrand is a single-valued function of $\lambda$ with a double pole at $\lambda = 0$ with residue† $t/\omega^2$.

Also there are simple poles at $\lambda = \pm i\omega$ with residues‡

$$\frac{e^{\pm i\omega t}}{\mp 2i\omega^3}\frac{J_0(\omega r/c)}{J_0(\omega a/c)}.$$

Finally there are simple poles at $\pm ic\alpha_s$, where $\pm\alpha_s$, $s = 1, 2,\ldots$, are the roots (all real and simple) of $J_0(az) = 0$. The residues at these are

$$\frac{e^{\pm ic\alpha_s t}J_0(r\alpha_s)}{\pm iac\alpha_s^2(\omega^2-c^2\alpha_s^2)J_0'(a\alpha_s)},$$

provided none of the $\alpha_s$ equals $\omega/c$ (in which case there is resonance with one of the natural frequencies and there are double poles at $\pm i\omega$). Collecting these terms, we have (from the usual argument, using Fig. 10) the solution

$$u = \frac{P_0\,c^2}{\omega^2 T}\sin\omega t\left(\frac{J_0(\omega r/c)}{J_0(\omega a/c)}-1\right)-\frac{2P_0\,\omega c}{aT}\sum_{s=1}^{\infty}\frac{\sin c\alpha_s t\,J_0(r\alpha_s)}{\alpha_s^2(\omega^2-c^2\alpha_s^2)J_0'(a\alpha_s)}.$$

† $I_0(z) = 1+\frac{z^2}{2^2}+\ldots$.

‡ $I_0(iz) = J_0(z)$, $I_0'(iz) = -iJ_0'(z)$.

**70.** *A circular membrane of radius a, stretched by tension T, is set in motion at $t = 0$ with velocity $f(r)$.*

The equation of motion is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad 0 \leqslant r < a,\, t > 0,$$

with $$u = 0, \quad \text{when } r = a,\, t > 0,$$

and $$u = 0, \qquad \frac{\partial u}{\partial t} = f(r), \quad \text{when } t = 0,\, 0 \leqslant r < a.$$

Writing $q = p/c$, the subsidiary equation becomes

$$\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}-q^2\bar{u} = -\frac{1}{c^2}f(r), \tag{1}$$

with $$\bar{u} = 0, \quad \text{when } r = a, \tag{2}$$

and $$\bar{u} \text{ to be finite}, \quad \text{when } r = 0. \tag{3}$$

To solve (1) we proceed as in § 68 and seek the Green's function† $G(r,\xi)$ for the homogeneous equation corresponding to (1) and the boundary conditions (2) and (3).

Assume

$$\left.\begin{aligned} G(r,\xi) &= AI_0(qr), \quad 0 \leqslant r \leqslant \xi, \\ &= BI_0(qr)+CK_0(qr), \quad \xi \leqslant r \leqslant a; \end{aligned}\right\} \tag{4}$$

this is finite when $r = 0$, and has in addition to be continuous at $r = \xi$, and to satisfy

$$\left.\begin{aligned} \left[\frac{\partial G}{\partial r}\right]_{r=\xi+0}-\left[\frac{\partial G}{\partial r}\right]_{r=\xi-0} &= 1 \\ \text{and} \qquad G(r,\xi) = 0, \quad \text{when } r &= a. \end{aligned}\right\}$$

These conditions require

$$BI_0(qa)+CK_0(qa) = 0,$$

$$AI_0(q\xi)-BI_0(q\xi)-CK_0(q\xi) = 0,$$

$$-AI_0'(q\xi)+BI_0'(q\xi)+CK_0'(q\xi) = \frac{1}{q\xi}.$$

† In Bromwich, *Proc. Lond. Math. Soc.* (2), **25** (1926), 103, the properties of the Green's function of this problem are fully discussed.

Solving and substituting in (4) we find

$$\left.\begin{aligned} G(r,\xi) &= -\frac{I_0(qr)}{I_0(qa)}\{I_0(qa)K_0(q\xi)-K_0(qa)I_0(q\xi)\}, \quad 0 \leqslant r \leqslant \xi,\\ &= -\frac{I_0(q\xi)}{I_0(qa)}\{I_0(qa)K_0(qr)-K_0(qa)I_0(qr)\}, \quad \xi \leqslant r \leqslant a.\end{aligned}\right\} \tag{5}$$

Then, proceeding as in § 68, the solution of (1) is found to be

$$\bar{u} = -\frac{1}{c^2}\int\limits_0^a \eta G(\eta,r)f(\eta)\,d\eta. \tag{6}$$

As an example, suppose $f(\eta)$ has the value $v/\epsilon^2$ if $0 < \eta < \epsilon$, where $\epsilon$ is small, and is zero outside this region. Then in the integral in (6) we may give $G(\eta,r)$ its value when $\eta = 0$ and obtain, using the first formula of (5) and the result $I_0(0) = 1$,

$$\bar{u} = \frac{v}{2c^2}\,\frac{I_0(qa)K_0(qr)-K_0(qa)I_0(qr)}{I_0(qa)}.$$

Thus, using the Inversion Theorem,

$$u = \frac{v\epsilon}{4i\pi c^2}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{I_0(a\lambda/c)K_0(r\lambda/c)-K_0(a\lambda/c)I_0(r\lambda/c)}{I_0(a\lambda/c)}\,d\lambda. \tag{7}$$

The integrand of (7) is a single-valued function of $\lambda$ with simple poles at $\lambda = \pm ic\alpha_s$, where $\alpha_s$, $s = 1, 2,\ldots$, are the roots of $J_0(a\alpha) = 0$.

The residue at the pole $\lambda = ic\alpha_s$ is

$$-e^{ict\alpha_s}\frac{cK_0(ia\alpha_s)I_0(ir\alpha_s)}{aI_0'(ia\alpha_s)} = +\frac{c}{ia^2}e^{ict\alpha_s}\frac{J_0(r\alpha_s)}{\alpha_s J_1^2(a\alpha_s)},$$

using § 68 (9).

Thus, using Fig. 10, it follows in the usual way that the line integral in (7) equals $2i\pi$ times the sum of the residues at the poles of its integrand, i.e.

$$u = \frac{v}{a^2c}\sum_{s=1}^{\infty}\frac{J_0(r\alpha_s)}{\alpha_s J_1^2(a\alpha_s)}\sin c\alpha_s t.$$

Similarly, we obtain from (6) for the general function $f(r)$

$$u = \frac{2}{a^2c}\sum_{s=1}\sin c\alpha_s t\,\frac{J_0(r\alpha_s)}{\alpha_s J_1^2(a\alpha_s)}\int\limits_0 \eta f(\eta)J_0(\eta\alpha_s)\,d\eta,$$

with suitable restrictions on $f(r)$.

# CHAPTER VIII

## HYDRODYNAMICS

**71.** *A long straight tube of cross-section $\alpha$ has at one point a close-fitting piston controlled by a spring, but otherwise free to move in the tube. The mass of the piston is $m$ and its period of oscillation* in vacuo *would be $2\pi/n$. The tube is open to the atmosphere at both ends and initially the piston and the air are at rest. A velocity $u$ is suddenly given to the piston at $t = 0$. Find the subsequent displacement of a layer of air at a distance $x$ from the piston. The equilibrium density of the air is $\rho_0$, and $c$ the velocity of sound.*†

Taking the equilibrium position of the piston as origin, let $\xi_1$ be the displacement of the layer of air at $x$ for $x > 0$, and $\xi_2$ that for $x < 0$; also let the displacement of the piston be $\xi$. Then

$$\xi_1 = \xi_2 = \xi, \quad \text{for } x = 0,\ t \geqslant 0. \tag{1}$$

Also $\xi_1$ and $\xi_2$ are to satisfy

$$\left.\begin{aligned} \frac{\partial^2 \xi_1}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \xi_1}{\partial t^2} &= 0, \quad t > 0,\ x > 0, \\ \frac{\partial^2 \xi_2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \xi_2}{\partial t^2} &= 0, \quad t > 0,\ x < 0. \end{aligned}\right\} \tag{2}$$

For the motion of the piston we have

$$m\frac{d^2\xi}{dt^2} + mn^2\xi = \rho_0 c^2 \alpha \left[\frac{\partial \xi_1}{\partial x} - \frac{\partial \xi_2}{\partial x}\right]_{x=0}, \quad t > 0. \tag{3}$$

These are to be solved with initial conditions

$$\frac{d\xi}{dt} = u,\ \xi = 0, \quad \text{when } t = 0,$$

$$\xi_1 = \frac{\partial \xi_1}{\partial t} = 0, \quad \text{when } t = 0,\ x > 0,$$

$$\xi_2 = \frac{\partial \xi_2}{\partial t} = 0, \quad \text{when } t = 0,\ x < 0.$$

† Math. Tripos, Part II, 1932.

The subsidiary equations corresponding to (2) are

$$\frac{d^2\bar{\xi}_1}{dx^2}-\frac{p^2}{c^2}\bar{\xi}_1 = 0, \quad x > 0, \tag{4}$$

$$\frac{d^2\bar{\xi}_2}{dx^2}-\frac{p^2}{c^2}\bar{\xi}_2 = 0, \quad x < 0. \tag{5}$$

The solutions of (4) and (5) which remain finite as $x \to \pm\infty$ respectively are

$$\bar{\xi}_1 = Ae^{-px/c}, \quad \text{and} \quad \bar{\xi}_2 = Be^{px/c}. \tag{6}$$

From (1) we have

$$\bar{\xi} = \lim_{x\to+0} \bar{\xi}_1 = \lim_{x\to-0} \bar{\xi}_2,$$

and thus from (6), $\qquad A = B = \bar{\xi}. \qquad (7)$

The subsidiary equation corresponding to (3) is

$$m(p^2+n^2)\bar{\xi} = mu+\rho_0 c^2\alpha\left[\frac{d\bar{\xi}_1}{dx}-\frac{d\bar{\xi}_2}{dx}\right]_{x=0}$$

and, on using (6) and (7), this becomes

$$\left(p^2+n^2+\frac{2\rho_0\alpha c}{m}p\right)\bar{\xi} = u. \tag{8}$$

Thus $$\bar{\xi} = \frac{u}{(p+k)^2+n^2-k^2},$$

and so $$\xi = \frac{u}{\sqrt{(n^2-k^2)}}e^{-kt}\sin t\sqrt{(n^2-k^2)},$$

where $k$ is written for $\rho_0\alpha c/m$, and we have assumed $k < n$.

Also, from (6), (7), and (8),

$$\bar{\xi}_1 = \frac{u}{(p+k)^2+n^2-k^2}e^{-px/c},$$

and therefore, by § 3, Theorem V,

$$\xi_1 = 0, \quad \text{when } t < x/c,$$
$$= \frac{u}{\sqrt{(n^2-k^2)}}e^{-k(t-x/c)}\sin(t-x/c)\sqrt{(n^2-k^2)}, \quad \text{when } t > x/c.$$

**72.** *A sphere of radius a, surrounded by air at rest, commences to pulsate radially at* $t = 0$.†

† Jeffreys, loc. cit., § 6.3. For other problems of the same type see Bromwich, *Proc. Lond. Math. Soc.* (2), **15** (1914), 431.

Let the displacement of the surface be

$$\xi = B\sin\omega t, \quad t > 0, \quad B \text{ supposed small.}$$

We have to solve

$$\frac{\partial^2(r\phi)}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2(r\phi)}{\partial t^2} = 0, \quad r > a,\, t > 0, \tag{1}$$

with $$-\frac{\partial\phi}{\partial r} \to \frac{d\xi}{dt} = \omega B\cos\omega t, \quad \text{as } r \to a, \text{ for } t > 0. \tag{2}$$

The subsidiary equation derived from (1) is

$$\frac{d^2(r\bar{\phi})}{dr^2}-\frac{p^2}{c^2}(r\bar{\phi}) = 0, \tag{3}$$

which by (2) has to be solved with

$$\frac{d\bar{\phi}}{dr} \to -\frac{Bp}{p^2+\omega^2}, \quad \text{as } r \to a. \tag{4}$$

The solution of (3), which is finite as $r \to \infty$, is $r\bar{\phi} = Ae^{-pr/c}$, and hence, by (4), we have

$$Ae^{-pa/c}\left[\frac{p}{ac}+\frac{1}{a^2}\right] = \frac{\omega Bp}{p^2+\omega^2}.$$

Thus, finally,

$$r\bar{\phi} = \frac{\omega Ba^2cp}{(p^2+\omega^2)(ap+c)}e^{-p(r-a)/c}$$

$$= \frac{\omega Ba^2c}{a^2\omega^2+c^2}\left[\frac{cp+\omega^2a}{p^2+\omega^2}-\frac{c}{p+c/a}\right]e^{-p(r-a)/c}$$

Therefore, by § 3, Theorem V,

$$r\phi = 0, \quad \text{when } t < \frac{r-a}{c},$$

$$= \frac{\omega Ba^2c^2}{a^2\omega^2+c^2}\left[\cos\omega\left(t-\frac{r-a}{c}\right)+\frac{\omega a}{c}\sin\omega\left(t-\frac{r-a}{c}\right)-e^{-\frac{c}{a}\left(t-\frac{r-a}{c}\right)}\right],$$

$$\text{when } t > \frac{r-a}{c}. \tag{5}$$

*As another example of the same type, suppose that the sphere only emits a single pulse.*

Then $$\left.\begin{aligned}\xi &= B\sin\omega t, \quad 0 \leqslant t \leqslant \pi/\omega, \\ &= 0, \quad t \geqslant \pi/\omega.\end{aligned}\right\}$$

Thus $$\bar{\xi} = B\int\limits_0^{\pi/\omega} e^{-pt}\sin\omega t\,dt = \frac{B\omega}{p^2+\omega^2}(1+e^{-p\pi/\omega}),$$

and proceeding as before we obtain

$$r\bar{\phi} = \frac{\omega Ba^2c}{(p^2+\omega^2)(ap+c)}e^{-p(r-a)/c}+\frac{\omega Ba^2c}{(p^2+\omega^2)(ap+c)}e^{-p\left(\frac{r-a}{c}+\frac{\pi}{\omega}\right)}.$$

The first term has been evaluated in (5), and for the second we have merely to replace $(r-a)/c$ in (5) by $(r-a)/c+\pi/\omega$, which gives

$$\frac{\omega Ba^2c^2}{a^2\omega^2+c^2}\times$$
$$\times\left[\cos\omega\left(t-\frac{r-a}{c}-\frac{\pi}{\omega}\right)+\frac{\omega a}{c}\sin\omega\left(t-\frac{r-a}{c}-\frac{\pi}{\omega}\right)-e^{\left(-\frac{ct-r+a}{a}+\frac{\pi c}{a\omega}\right)}\right]$$
$$=\frac{\omega Ba^2c^2}{a^2\omega^2+c^2}\times$$
$$\times\left[-\cos\omega\left(t-\frac{r-a}{c}\right)-\frac{\omega a}{c}\sin\omega\left(t-\frac{r-a}{c}\right)-e^{\left(-\frac{ct-r+a}{a}+\frac{\pi c}{a\omega}\right)}\right],$$

when $t > (r-a)/c+\pi/\omega$, and 0 when $t < (r-a)/c+\pi/\omega$.

Adding these results, we have finally

$$r\phi = 0,\quad \text{when } t < \frac{r-a}{c},$$

$$= \frac{\omega Ba^2c^2}{a^2\omega^2+c^2}\left[\cos\omega\left(t-\frac{r-a}{c}\right)+\frac{\omega a}{c}\sin\omega\left(t-\frac{r-a}{c}\right)-e^{-(ct-r+a)/a}\right],$$
$$\text{when } \frac{r-a}{c} < t < \frac{r-a}{c}+\frac{\pi}{\omega},$$

$$= -\frac{\omega Ba^2c^2}{a^2\omega^2+c^2}e^{-(ct-r+a)/a}[1+e^{\pi c/\omega a}],\quad \text{when } t > \frac{r-a}{c}+\frac{\pi}{\omega}.$$

**73.** *A sphere of radius $b$ and mass $M$ makes small linear oscillations in air of density $\rho$ under a restoring force $Mn^2$ times the displacement.*† *The motion is started from rest in the equilibrium position by giving the sphere a velocity $U$ at $t = 0$.*

† Love, *Proc. Lond. Math. Soc.* (2), **2** (1904), 100; Bromwich, ibid. (2), **15** (1914), 431; Lamb, *Hydrodynamics* (5th ed., 1924), § 301.

Let $\xi$ be the displacement of the centre of the sphere at time $t$. Then for small motions the surface of the sphere may be taken to be

$$r = b+\xi\cos\theta, \tag{1}$$

where $\theta$ is measured from the line of motion as axis.

Because of the form of (1) we seek a velocity potential in the air of type $\phi = R\cos\theta$, where $R$ is independent of $\theta$; then continuity of normal velocity at the surface of the sphere requires

$$\frac{d\xi}{dt} = -\left[\frac{\partial R}{\partial r}\right]_{r=b}. \tag{2}$$

The equation of motion of the sphere is

$$\frac{d^2\xi}{dt^2}+n^2\xi = -\frac{\rho}{M}\left[\frac{\partial R}{\partial t}\right]_{r=b}\int\limits_0^\pi \cos^2\theta\, 2\pi b^2\sin\theta\, d\theta = -\frac{\beta}{b}\left[\frac{\partial R}{\partial t}\right]_{r=b} \tag{3}$$

where $\beta = 4\pi\rho b^3/3M = \rho/\sigma$, and $\sigma$ is the density of the sphere.

Also, since $\phi$ satisfies $\nabla^2\phi-\dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} = 0$, $R$ must satisfy

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2R}{r^2}-\frac{1}{c^2}\frac{\partial^2 R}{\partial t^2} = 0. \tag{4}$$

The subsidiary equations corresponding to (4), (2), and (3) are

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\bar{R}}{dr}\right)-\left(\frac{2}{r^2}+\frac{p^2}{c^2}\right)\bar{R} = 0, \tag{5}$$

$$p\bar{\xi} = -\left[\frac{d\bar{R}}{dr}\right]_{r=b}, \tag{6}$$

$$(p^2+n^2)\bar{\xi} = U-(\beta p/b)[\bar{R}]_{r=b}. \tag{7}$$

To solve (5) put $\bar{R} = Yr^{-\frac{1}{2}}$. Then $Y$ satisfies

$$\frac{d^2Y}{dr^2}+\frac{1}{r}\frac{dY}{dr}-\left(\frac{9}{4r^2}+\frac{p^2}{c^2}\right)Y = 0.$$

The solution of this, which remains finite as $r\to\infty$, is

$$Y = AK_{\frac{3}{2}}(pr/c).$$

Thus

$$\bar{R} = Ar^{-\frac{1}{2}}K_{\frac{3}{2}}(pr/c)$$

$$= A\sqrt{\left(\frac{\pi c}{2p}\right)}\left(\frac{1}{r}+\frac{c}{pr^2}\right)e^{-pr/c}, \tag{8}$$

where we have used the result $K_{\frac{3}{2}}(z) = \sqrt{(\pi/2z)}e^{-z}(1+1/z)$.

To find $A$ we substitute (8) in (6) and (7) and obtain on reduction

$$A = \sqrt{\left(\frac{2p}{\pi c}\right)} e^{pb/c} \frac{p^2 cb^3 U}{(p^2+n^2)(p^2b^2+2pbc+2c^2)+\beta p^2 c(pb+c)}.$$

Therefore

$$\bar{\phi} = \bar{R}\cos\theta$$

$$b^3 cU\cos\theta \frac{p(pr+c)}{(p^2+n^2)(p^2b^2+2pbc+2c^2)+\beta cp^2(pb+c)} e^{-p(r-b)/c}. \tag{9}$$

$\phi$ may be found from (9) when the roots of the biquadratic in the denominator are known. An approximate solution is obtained by neglecting $\beta$, the ratio of the mass of the sphere to that of the air it displaces, which in practice is small. Then, putting $\beta = 0$ and $\alpha = c/b$ in (9), and expressing the right-hand side in partial fractions, we have

$$\frac{r^2(n^4+4\alpha^4)}{bcU\cos\theta}\bar{\phi}$$

$$= \frac{[2\alpha n^2 r - c(n^2-2\alpha^2)]p+[2n^2\alpha c+r(n^4-2\alpha^2 n^2)]}{p^2+n^2} e^{-p(r-b)/c}+$$

$$+\frac{[c(n^2-2\alpha^2)-2\alpha n^2 r]p+[r(4\alpha^4-2\alpha^2 n^2)-4\alpha^3 c]}{(p+\alpha)^2+\alpha^2} e^{-p(r-b)/c}.$$

Thus, using § 3, Theorems IV and V, we have

$$\frac{r^2(n^4+4\alpha^4)}{bcU\cos\theta}\phi = [2\alpha n^2 r - c(n^2-2\alpha^2)]\cos n\left(t-\frac{r-b}{c}\right)+$$

$$+[nr(n^2-2\alpha^2)+2n\alpha c]\sin n\left(t-\frac{r-b}{c}\right)+$$

$$+[c(n^2-2\alpha^2)-2\alpha n^2 r]e^{-\alpha[t-(r-b/c)]}\cos\alpha\left(t-\frac{r-b}{c}\right)+$$

$$+[r(4\alpha^3-2\alpha n^2)-4\alpha^2 c]e^{-\alpha[t-(r-b/c]}\sin\alpha\left(t-\frac{r-b}{c}\right.$$

when $t > r-b$

$= 0$, when $t < r-b$

**74.** *A right circular cylinder of radius a contains air. The whole is moving with velocity V perpendicular to its length when the cylinder is stopped at $t = 0$. To find the subsequent motion of the air.*

If $\phi$ is the velocity potential, we have to solve

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \quad t > 0,\ r < a, \tag{1}$$

with
$$\phi = -Vr\cos\theta, \quad t = 0,\ r < a, \tag{2}$$

and
$$\frac{\partial\phi}{\partial r} = 0, \quad r = a,\ t > 0. \tag{3}$$

The subsidiary equation is

$$\nabla^2\bar{\phi} - \frac{p^2}{c^2}\bar{\phi} = \frac{p}{c^2}Vr\cos\theta, \tag{4}$$

with
$$\frac{\partial\bar{\phi}}{\partial r} = 0, \quad \text{when } r = a. \tag{5}$$

We seek a solution of (4) of type $\bar{\phi} = R(r)\cos\theta$. Then $R$ has to satisfy

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\frac{1}{r^2} + \frac{p^2}{c^2}\right)R = \frac{pV}{c^2}r.$$

A particular integral of this is $-Vr/p$, and a solution, finite at the origin, of the corresponding homogeneous equation is $AI_1(pr/c)$. Thus the general solution of (4) is

$$\bar{\phi} = AI_1\left(\frac{pr}{c}\right)\cos\theta - \frac{Vr}{p}\cos\theta,$$

and the condition (5) requires

$$\frac{Ap}{c}I_1'\left(\frac{pa}{c}\right) - \frac{V}{p} = 0.$$

Therefore
$$\bar{\phi} = -\frac{Vr}{p}\cos\theta + \frac{cV}{p^2}\frac{I_1(pr/c)}{I_1'(pa/c)}\cos\theta.$$

Thus, using the Inversion Theorem,

$$\phi = -Vr\cos\theta + \frac{cV\cos\theta}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{I_1(\lambda r/c)}{I_1'(\lambda a/c)}\frac{d\lambda}{\lambda^2}. \tag{6}$$

Using the series $I_1(z) = \frac{1}{2}z(1+\frac{1}{8}z^2+...)$, we see that the integrand in (6) is a single-valued function of $\lambda$ with simple poles at $\lambda = 0$ and $\lambda = \pm ic\alpha_s$, where $\pm\alpha_s$, $s = 1, 2,...,$ are the roots† of

$$J_1'(a\alpha) = 0.$$

We use the contour of Fig. 10, choosing the radius of the circle as $n\pi c/a$ so that it will not pass through any zero of $I_1'(\lambda a/c)$. From the asymptotic expansions of the Bessel functions it can be shown that when $n \to \infty$ the integral over the circle $BCA$ tends to zero. Thus the integral in (6) may be replaced by the limit of the integral over the closed circuit of Fig. 10 as $n \to \infty$.

The residue of the integrand of (6) at the pole $\lambda = 0$ is $r/c$, while that at $\lambda = ic\alpha_s$ is

$$\frac{1}{ac\alpha_s^2} e^{ict\alpha_s} \frac{J_1(\alpha_s r)}{J_1''(a\alpha_s)}.$$

Thus, finally, by Cauchy's theorem,

$$\phi = \frac{2V\cos\theta}{a} \sum_{s=1}^{\infty} \frac{J_1(r\alpha_s)}{\alpha_s^2 J_1''(a\alpha_s)} \cos\alpha_s ct.$$

**75.** *Viscous fluid between parallel planes $y = \pm h$ is set in motion by uniform body force $X$ applied at $t = 0$.*‡

If $u$ is the velocity parallel to $X$, and $\nu$ the kinematic viscosity, we have to solve

$$\frac{\partial u}{\partial t} = X+\nu\frac{\partial^2 u}{\partial y^2}, \quad -h < y < h, \quad t > 0,$$

with $u = 0$ when $y = \pm h$, $t > 0$.

The subsidiary equation is

$$\frac{d^2\bar{u}}{dy^2}-\frac{p}{\nu}\bar{u} = -\frac{X}{p\nu},$$

to be solved with $\bar{u} = 0$ when $y = \pm h$.

The solution is

$$\bar{u} = \frac{X}{p^2}\left[1-\frac{\cosh y\sqrt{(p/\nu)}}{\cosh h\sqrt{(p/\nu)}}\right].$$

† $I_1(iz) = iJ_1(z)$.

‡ Bromwich, *Journ. Lond. Math. Soc.* **5** (1930), 10.

Thus, by the Inversion Theorem,

$$u = \frac{X}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\left[1-\frac{\cosh y\sqrt{(\lambda/\nu)}}{\cosh h\sqrt{(\lambda/\nu)}}\right]\frac{d\lambda}{\lambda^2}. \tag{1}$$

The integrand of (1) is a single-valued function of $\lambda$ with simple poles at $\lambda = 0$ and $\lambda = -(\nu\pi^2/h^2)(n+\frac{1}{2})^2$, $n = 0, 1, 2,\ldots$. Using the contour of Fig. 10 we may justify in the usual way the replacing of the integral in (1) by $2i\pi$ times the sum of the residues at poles within the closed circuit.

The residue at $\lambda = 0$ is

$$\frac{h^2-y^2}{2\nu}.$$

The residue at $\lambda = -(\nu\pi^2/h^2)(n+\frac{1}{2})^2$ is

$$-\frac{16h^2}{\nu\pi^3}\frac{e^{-(\nu\pi^2/h^2)(n+\frac{1}{2})^2t}\cos\{(2n+1)\pi/2h\}y}{(2n+1)^3\sin(n+\frac{1}{2})\pi}.$$

So, finally,

$$u = \frac{X(h^2-y^2)}{2\nu}-\frac{16h^2X}{\nu\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\frac{(2n+1)y}{2h}e^{-(\nu\pi^2/h^2)(n+\frac{1}{2})^2t}.$$

**76.** *A heavy vertical thin lamina falls under gravity, from rest at $t = 0$, through viscous liquid between parallel vertical walls*† *(body force on the liquid neglected).*

Let $\sigma$ be the mass per unit area of the plate, $\rho$ the density of the liquid, $\nu$ the kinematic viscosity, $v$, $V$ the velocities of the liquid and plate respectively, and $h$ the distance of the plate from the walls.

Then for the motion of the liquid we have

$$\frac{\partial v}{\partial t} = \nu\frac{\partial^2 v}{\partial x^2}, \quad -h < x < 0 \text{ and } 0 < x < h,\ t > 0,$$

with

$$\begin{aligned} &v = 0, \quad x = \pm h, \quad t > 0, \\ &v = V, \quad x = 0, \quad t > 0, \\ &v = V = 0, \quad t = 0; \end{aligned} \tag{1}$$

† Crossley, *Proc. Camb. Phil. Soc.* **24** (1928), 231.

while for the motion of the plate we have

$$\frac{dV}{dt} = g+\frac{2\rho\nu}{\sigma}\left[\frac{\partial v}{\partial x}\right]_{x=0}. \tag{2}$$

The subsidiary equation derived from (1) is

$$\frac{d^2\bar{v}}{dx^2}-\frac{p}{\nu}\bar{v} = 0,$$

with $\bar{v} = 0$ when $x = \pm h$, and $\bar{v} = \bar{V}$ when $x = 0$.

The solution of these is

$$\bar{v} = \bar{V}\frac{\sinh(h-x)\sqrt{(p/\nu)}}{\sinh h\sqrt{(p/\nu)}}, \quad 0 < x < h. \tag{3}$$

The subsidiary equation corresponding to (2) is

$$p\bar{V} = \frac{g}{p}+\frac{2\rho\nu}{\sigma}\left[\frac{d\bar{v}}{dx}\right]_{x=0},$$

and introducing the value of $\bar{v}$ above we have

$$\bar{V} = \frac{g}{p}\left[p+\frac{2\rho\nu}{\sigma}\sqrt{(p/\nu)}\coth h\sqrt{(p/\nu)}\right]^{-1}. \tag{4}$$

Thus, using the Inversion Theorem,

$$= \frac{g}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda}\frac{d\lambda}{\{\lambda+(2\rho\nu/\sigma)\sqrt{(\lambda/\nu)}\coth h\sqrt{(\lambda/\nu)}\}}. \tag{5}$$

The integrand is a single-valued function of $\lambda$ with a simple pole at $\lambda = 0$, which gives a term $\sigma gh/2\rho\nu$.

To find the other poles put $\lambda = -(\nu/h^2)\alpha^2$ in the denominator which becomes

$$\frac{\nu^2}{h^4}\alpha^3\cot\alpha\{\alpha\tan\alpha-k\}, \quad \text{where } k = \frac{2\rho h}{\sigma}.$$

So the poles of the integrand (other than $\lambda = 0$) are at $\lambda = -(\nu/h^2)\alpha_s^2$, $s = 1, 2, 3,\ldots$, where the $\alpha_s$ are the real positive roots of $\alpha\tan\alpha-k = 0$.

Then, using the contour of Fig. 10 and proceeding in the usual way, we find†

$$V = \frac{\sigma gh}{2\rho\nu}-\frac{4g\rho h^3}{\nu\sigma}\sum_{s=1}^{\infty}\frac{e^{-\nu t\alpha_s^2/h^2}}{\alpha_s^2(\alpha_s^2+k^2+k)}.$$

† $\left[\frac{d}{d\lambda}\lambda\left\{\lambda+\frac{2\rho\nu}{\sigma}\sqrt{(\lambda/\nu)}\coth\sqrt{(\lambda/\nu)}h\right\}\right]_{\lambda=-(\nu/h^2)\alpha_s^2} = -\frac{\nu\sigma}{4\rho h^3}\alpha_s^2(\alpha_s^2+k^2+k).$

For the velocity of the liquid we have, from (3) and (4),

$$\bar{v} = \frac{g \sinh(h-x)\sqrt{(p/\nu)}}{p\{p+(2\rho\nu/\sigma)\sqrt{(p/\nu)}\coth h\sqrt{(p/\nu)}\}\sinh h\sqrt{(p/\nu)}},$$

and evaluating as before (the poles of the integrand are the same as those in (5))

$$v = \frac{gh\sigma}{2\rho\nu}\frac{h-x}{h} - \frac{4\rho gh^3}{\nu\sigma}\sum_{s=1}^{\infty}\frac{e^{-\nu t\alpha_s^2/h^2}}{\alpha_s^2(\alpha_s^2+k^2+k)}\frac{\sin\alpha_s(h-x)/h}{\sin\alpha_s},$$
$$0 < x < h.$$

**77.** *Viscous fluid is contained between two infinite concentric cylinders,*† *radii $a$ and $b$. At $t = 0$ the outer cylinder starts rotating with angular velocity $\Omega$. To find the subsequent motion of the liquid.*

We have to solve

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2} = \frac{1}{\nu}\frac{\partial v}{\partial t}, \quad a < r < b,\ t > 0,$$

with $v = \Omega b$ when $r = b$, and $v = 0$ when $r = a$.

The subsidiary equation is

$$\frac{d^2\bar{v}}{dr^2}+\frac{1}{r}\frac{d\bar{v}}{dr}-\left(\frac{1}{r^2}+q^2\right)\bar{v} = 0, \quad \text{where } q^2 = p/\nu,$$

the solution of which is

$$\bar{v} = AI_1(qr)+BK_1(qr).$$

To determine $A$ and $B$ we know that $\bar{v} = 0$ when $r = a$, and $\bar{v} = \Omega b/p$ when $r = b$. So

$$AI_1(qa)+BK_1(qa) = 0,$$
$$AI_1(qb)+BK_1(qb) = \Omega b/p.$$

Solving, we have finally

$$\bar{v} = \frac{\Omega b}{p}\frac{I_1(qa)K_1(qr)-K_1(qa)I_1(qr)}{I_1(qa)K_1(qb)-K_1(qa)I_1(qb)}.$$

† Goldstein, *Proc. Lond. Math. Soc.* (2), **34** (1931), 51. This important paper contains a large number of problems on viscous motion, conduction of heat, and diffusion of vorticity. The last of these subjects is not discussed in this chapter as the problems are similar to those of Conduction of Heat.

Therefore, by the Inversion Theorem,

$$v = \frac{\Omega b}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{I_1\{a\sqrt{(\lambda/\nu)}\}K_1\{r\sqrt{(\lambda/\nu)}\}-K_1\{a\sqrt{(\lambda/\nu)}\}I_1\{r\sqrt{(\lambda/\nu)}\}}{I_1\{a\sqrt{(\lambda/\nu)}\}K_1\{b\sqrt{(\lambda/\nu)}\}-K_1\{a\sqrt{(\lambda/\nu)}\}I_1\{b\sqrt{(\lambda/\nu)}\}}\frac{e^{\lambda t}\,d\lambda}{\lambda}. \tag{1}$$

It is easily verified that the integrand is a single-valued function of $\lambda$, so we use the contour of Fig. 10. To find the zeros of the denominator put $\lambda = -\nu\alpha^2$, then† it becomes

$$I_1(ia\alpha)K_1(ib\alpha)-K_1(ia\alpha)I_1(ib\alpha) = -\tfrac{1}{2}\pi\{J_1(a\alpha)Y_1(b\alpha)-Y_1(a\alpha)J_1(b\alpha)\}.$$

Let $\alpha_1$, $\alpha_2$,... be the roots of the equation‡

$$J_1(a\alpha)Y_1(b\alpha)-J_1(ba)Y_1(a\alpha) = 0. \tag{2}$$

Then the poles of the integrand of (1) are $\lambda = 0$ and $\lambda = -\nu\alpha_1^2$, $-\nu\alpha_2^2$,.... But

$$\frac{d}{d\alpha}[J_1(a\alpha)Y_1(b\alpha)-Y_1(a\alpha)J_1(ba)] = aJ_1'(a\alpha)Y_1(b\alpha)+bJ_1(a\alpha)Y_1'(b\alpha)- -aY_1'(a\alpha)J_1(b\alpha)-bY_1(a\alpha)J_1'(b\alpha).$$

Also, if $\alpha_1$ is a root of (2),

$$\frac{J_1(a\alpha_1)}{J_1(b\alpha_1)} = \frac{Y_1(a\alpha_1)}{Y_1(b\alpha_1)} = k, \quad \text{say.}$$

Therefore

$$\left[\frac{d}{d\alpha}\{J_1(a\alpha)Y_1(b\alpha)-Y_1(a\alpha)J_1(b\alpha)\}\right]_{\alpha=\alpha_1}$$
$$= bk\{J_1(b\alpha_1)Y_1'(b\alpha_1)-J_1'(b\alpha_1)Y_1(b\alpha_1)\}- -\frac{a}{k}\{J_1(a\alpha_1)Y_1'(a\alpha_1)-J_1'(a\alpha_1)Y_1(a\alpha_1)\}$$
$$= \frac{2}{\pi\alpha_1}\left(k-\frac{1}{k}\right),\|$$
$$= \frac{2}{\pi\alpha_1}\,\frac{J_1^2(a\alpha_1)-J_1^2(b\alpha_1)}{J_1(a\alpha_1)J_1(b\alpha_1)}.$$

So the residue at the pole $\lambda = -\nu\alpha_1^2$ is

$$\pi\frac{[J_1(a\alpha_1)Y_1(r\alpha_1)-Y_1(a\alpha_1)J_1(r\alpha_1)]}{[J_1^2(a\alpha_1)-J_1^2(b\alpha_1)]}J_1(a\alpha_1)J_1(b\alpha_1)e^{-\nu\alpha_1^2 t}.$$

† $I_1(iz) = iJ_1(z)$, $K_1(iz) = -\frac{1}{2}\pi[J_1(z)-iY_1(z)]$.

‡ It is known that these are all real and simple. Cf. *G. and M.*, p. 82.

‖ $J_1(z)Y_1'(z)-Y_1(z)J_1'(z) = 2/\pi z$.

Also, since $I_1(z) = \frac{1}{2}z\{1+\frac{1}{8}z^2+...\}$ and $K_1(z) = 1/z+\frac{1}{2}z\log\frac{1}{2}z+...$, the residue at $\lambda = 0$ is $\dfrac{b}{r}\dfrac{r^2-a^2}{b^2-a^2}$.

Thus, finally

$$v = \frac{\Omega b^2}{r}\frac{r^2-a^2}{b^2-a^2}-\pi b\Omega\sum_{s=1}^{\infty}\frac{J_1(a\alpha_s)Y_1(r\alpha_s)-Y_1(a\alpha_s)J_1(r\alpha_s)}{J_1^2(b\alpha_s)-J_1^2(a\alpha_s)}J_1(b\alpha_s)J_1(a\alpha_s)e^{-\nu\alpha_s^2t}.$$

**78.** *A cylinder of radius a and moment of inertia I, per unit length, immersed in infinite viscous liquid is set in motion by a couple N, per unit length, applied at* $t = 0$.

For the motion of the liquid† we have

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2} = \frac{1}{\nu}\frac{\partial v}{\partial t}, \quad t > 0,\ r > a, \tag{1}$$

with $v \to 0$, as $r \to \infty$, and $v = V$ when $r = a$, where $V$ is the peripheral velocity of the cylinder.

The frictional couple per unit length of the cylinder is

$$\left[2\pi r\, r\,\rho\nu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\right]_{r=a}.$$

Therefore the equation of motion of the cylinder is

$$\frac{I}{a}\frac{dV}{dt} = N+2\pi a^2\rho\nu\left[\frac{\partial v}{\partial r}-\frac{v}{r}\right]_{r=a}. \tag{2}$$

The subsidiary equation corresponding to (1) is

$$\frac{d^2\bar{v}}{dr^2}+\frac{1}{r}\frac{d\bar{v}}{dr}-\left(\frac{1}{r^2}+q^2\right)\bar{v} = 0, \quad \text{where } q^2 = p/\nu,$$

with $\bar{v} = \bar{V}$ when $r = a$.

The solution of this which remains finite as $r \to \infty$ is

$$\bar{v} = \bar{V}\frac{K_1(qr)}{K_1(qa)}. \tag{3}$$

The subsidiary equation derived from (2) is

$$\frac{I}{a}p\bar{V} = \frac{N}{p}+2\pi a^2\rho\nu\left[\frac{d\bar{v}}{dr}-\frac{\bar{v}}{r}\right]_{r=a},$$

† End effects are neglected; i.e. the motion is taken as two-dimensional.

and substituting for $\bar{v}$ from (3) in this we have†

$$\bar{V} = \frac{Na^3}{\nu I}\frac{K_1(aq)}{apq[aqK_1(aq)+kK_2(aq)]}, \quad \text{where } k \; 2\pi a^4 \rho$$

Thus, using the Inversion Theorem,

$$V = \frac{1}{2\pi i}\frac{Na^3}{\nu I}\int\limits_{\gamma-i}^{\gamma+i\infty} \frac{e^{\lambda t}K_1\{a\sqrt{(\lambda/\nu)}\}\,d\lambda}{\lambda a\sqrt{(\lambda/\nu)}[a\sqrt{(\lambda/\nu)}K_1\{a\sqrt{(\lambda/\nu)}\}+kK_2\{a\sqrt{(\lambda/\nu)}\}]}. \tag{4}$$

The integrand of (4) has a branch-point at $\lambda = 0$, so the contour of Fig. 11 must be used. It is easily verified that there are no poles in or on this contour, and that the integral over the large circle vanishes as its radius tends to infinity.

The integral round the small circle gives‡

$$\frac{Na^3}{2\nu kI} = \frac{N}{4\pi a\rho\nu}.$$

Putting $\lambda = \nu u^2 e^{-i\pi}$ on $CD$, $\lambda = \nu u^2 e^{i\pi}$ on $EF$, and using the relations

$$K_1(iz) = -\tfrac{1}{2}\pi[J_1(z)-iY_1(z)], \qquad K_2(iz) = \tfrac{1}{2}\pi i[J_2(z)-iY_2(z)],$$

we obtain after some reduction

$$V = \frac{N}{4\pi a\rho\nu}+$$

$$+\frac{2Na^2k}{\pi\nu I}\int^{\infty} \frac{e^{-\nu u^2 t}}{u^2}\frac{J_1(au)Y_2(au)-Y_1(au)J_2(au)}{[auJ_1(au)-kJ_2(au)]^2+[auY_1(au)-kY_2(au)]^2}\,du.$$

**79.** *A canal of rectangular section, containing water to a mean depth h, is terminated by two vertical walls whose distance apart is 2l. The water is initially at rest with its surface inclined at a small angle β to the horizontal. It is required to find the subsequent form of the surface.*

I. *Neglect the vertical acceleration of particles of the liquid.*

Let $\eta$ be the displacement at $x$ of the surface above its

† $zK_1'(z)-K_1(z) = -zK_2(z)$.

‡ $K_1(z) = \frac{1}{z}+\frac{1}{2}z\log\frac{1}{2}z+\ldots$, $\quad K_2(z) = \frac{2}{z^2}-\frac{1}{2}-\ldots$.

equilibrium level, the origin of $x$ being taken at the middle point of the length of the canal.

Then, writing $c^2 = \sqrt{(gh)}$, we have to solve

$$\frac{\partial^2\eta}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2} = 0, \quad -l < x < l,\ t > 0,$$

with $\partial\eta/\partial x = 0$ when $x = \pm l$.

Since the initial value of $\eta$ is $\beta x$, and that of $\partial\eta/\partial t$ is zero, the subsidiary equation is

$$\frac{d^2\bar{\eta}}{dx^2} - \frac{p^2}{c^2}\bar{\eta} = -\frac{\beta xp}{c^2},$$

with $d\bar{\eta}/dx = 0$ when $x = \pm l$.

The solution is

$$\bar{\eta} = \frac{\beta x}{p} - \frac{\beta c \sinh(px/c)}{p^2 \cosh(pl/c)}.$$

Thus, using the Inversion Theorem,

$$\eta = \beta x - \frac{\beta c}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda^2}\frac{\sinh(\lambda x/c)}{\cosh(\lambda l/c)}\,d\lambda.$$

The integrand is a single-valued function of $\lambda$, so we use the contour of Fig. 10.

In the usual way we find that the integral is $2\pi i$ times the sum of the residues at the poles of the integrand. These poles (all simple) are at $\lambda = 0$ and $\lambda = (2n+1)i\pi c/2l$, $n = 0, \pm 1, \ldots$.

The pole at $\lambda = 0$ has residue $x/c$, while that at $(2n+1)i\pi c/2l$ has residue

$$-\frac{4l(-1)^n}{(2n+1)^2\pi^2 c}e^{(2n+1)\pi i ct/2l}\sin\frac{(2n+1)\pi x}{2l}.$$

Thus, finally,

$$\eta = \frac{8\beta l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\cos(n+\tfrac{1}{2})\frac{\pi ct}{l}\sin(n+\tfrac{1}{2})\frac{\pi x}{l}.$$

II. *The canal is too deep for the assumption* I *to be valid.*

We take the $x$-axis in the equilibrium position of the free

surface, and the $y$-axis vertically upwards. Then, if $\phi$ is the velocity potential, we have to solve

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0, \quad \text{when } -h < y < 0,\ -l < x < l,\ t > 0,$$

$$\frac{\partial\phi}{\partial y} = 0, \quad \text{when } y = -h,\ -l < x < l,\ t > 0,$$

$$\frac{\partial\phi}{\partial x} = 0, \quad \text{when } x = \pm l,\ -h < y < 0,\ t > 0,$$

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial y} = 0, \quad \text{when } y = 0,\ -l < x < l,\ t > 0,$$

with the initial conditions

$$\phi = 0, \quad \text{when } -l < x < l,\ -h < y < 0,$$

and
$$\frac{\partial\phi}{\partial t} = g\beta x, \quad \text{when } y = 0,\ -l < x < l.$$

The subsidiary equations are

$$\frac{\partial^2\bar\phi}{\partial x^2}+\frac{\partial^2\bar\phi}{\partial y^2} = 0, \quad -h < y < 0,\ -l < x < l, \tag{1}$$

with
$$\frac{\partial\bar\phi}{\partial y} = 0, \quad \text{when } y = -h,\ -l < x < l, \tag{2}$$

$$\frac{\partial\bar\phi}{\partial x} = 0, \quad \text{when } x = \pm l,\ -h < y < 0, \tag{3}$$

and
$$p^2\bar\phi+g\frac{\partial\bar\phi}{\partial y} = g\beta x, \quad \text{when } y = 0,\ -l < x < l. \tag{4}$$

A solution of (1) satisfying (2) and (3) is

$$\cosh\frac{\pi(2n+1)(y+h)}{2l}\sin\frac{(2n+1)\pi x}{2l}, \quad n = 0, 1, 2,\dots.$$

So we assume

$$\bar\phi = \sum_{n=0}^{\infty} A_n \cosh\frac{\pi(2n+1)(y+h)}{2l}\sin\frac{(2n+1)\pi x}{2l}. \tag{5}$$

Substituting in (4) we have

$$\sum_{n=0}^{\infty} A_n\Big[p^2\cosh\frac{\pi h(2n+1)}{2l}+$$
$$+\frac{\pi g(2n+1)}{2l}\sinh\frac{\pi h(2n+1)}{2l}\Big]\sin\frac{(2n+1)\pi x}{2l}$$
$$= g\beta x$$
$$= \frac{8g\beta l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l},$$

introducing in the last line the Fourier series for $x$.

Thus

$$A_n = \frac{8g\beta l}{\pi^2}\frac{(-1)^n}{(2n+1)^2}\Big[p^2\cosh\frac{\pi h(2n+1)}{2l}+$$
$$+\frac{\pi g(2n+1)}{2l}\sinh\frac{\pi h(2n+1)}{2l}\Big]^{-1},$$

and substituting this in (5) we have

$$\bar{\phi} = \frac{8g\beta l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\cosh\{\pi(2n+1)(y+h)/2l\}\sin\{(2n+1)\pi x/2l\}}{(2n+1)^2(p^2+\omega_n^2)\cosh\{\pi h(2n+1)/2l\}}, \tag{6}$$

where
$$\omega_n^2 = \frac{\pi g(2n+1)}{2l}\tanh\frac{\pi h(2n+1)}{2l}. \tag{7}$$

$\phi$ may be determined from (6). We require the surface elevation $\eta$ which is given by

$$\eta = \frac{1}{g}\Big[\frac{\partial\phi}{\partial t}\Big]_{y=0}.$$

And thus

$$\bar{\eta} = \frac{p}{g}[\bar{\phi}]_{y=0}$$
$$= \frac{8\beta l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{p}{(p^2+\omega_n^2)}\sin\frac{(2n+1)\pi x}{2l}.$$

Therefore

$$\eta = \frac{8\beta l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\cos\omega_n t\sin\frac{(2n+1)\pi x}{2l}.$$

**80.** *Deep water waves† in infinite liquid produced by an initial surface elevation.*

Let the $x$-axis be in the free surface of the liquid, with the $y$-axis vertically upwards; the surface elevation is taken to be a function of $x$ only.

We have to solve

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0, \quad y < 0, -\infty < x < \infty, t > 0,$$

with
$$\frac{\partial\phi}{\partial y} \to 0 \quad \text{as} \quad y \to -\infty,$$

and

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial y} = 0, \quad \text{when } y = 0, -\infty < x < \infty, t > 0.$$

The initial conditions are

$$\phi = 0, \quad \text{when } y < 0, -\infty < x < \infty,$$

and
$$\frac{\partial\phi}{\partial t} = gf(x), \quad \text{when } y = 0, -\infty < x < \infty,$$

where $f(x)$ is the (small) initial surface elevation.

The subsidiary equations are

$$p^2\bar{\phi}+g\frac{\partial\bar{\phi}}{\partial y} = gf(x), \quad y = 0, -\infty < x < \infty, \tag{1}$$

$$\frac{\partial^2\bar{\phi}}{\partial x^2}+\frac{\partial^2\bar{\phi}}{\partial y^2} = 0, \quad y < 0, -\infty < x < \infty, \tag{2}$$

with
$$\frac{\partial\bar{\phi}}{\partial y} \to 0 \quad \text{as} \quad y \to -\infty. \tag{3}$$

We take as a general solution of (2) and (3)

$$\bar{\phi} = \int\limits_{-\infty}^{\infty} \psi(m)e^{|m|y+imx}\, dm.$$

Substituting in (1) we obtain

$$\int\limits^{\infty} (p^2+|m|g)\psi(m)e^{imx}\, dm = gf(x), \tag{4}$$

† Lamb, loc. cit., § 238.

and, inverting this by Fourier's Integral Theorem,† we have

$$\psi(m) = \frac{g}{2\pi}\frac{1}{p^2+|m|g}\int\limits_{-\infty}^{\infty} f(x')e^{-imx'}\,dx'. \tag{5}$$

So $$\bar{\phi} = \frac{g}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{|m|y+imx}\,dm}{p^2+g|m|}\int f(x')e^{-imx'}\,dx'.$$

Therefore

$$\phi = \frac{g}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\sin\sqrt{(|m|g)}t}{\sqrt{(|m|g)}}e^{|m|y+imx}\,dm\int\limits_{-\infty}^{\infty} f(x')e^{-imx'}\,dx'$$

$$= \frac{g}{\pi}\int\limits_{0}^{\infty}\frac{\sin t\sqrt{(gk)}}{\sqrt{(gk)}}e^{ky}\,dk\int\limits_{-\infty}^{\infty} f(x')\cos k(x-x')\,dx'.$$

**81.** *Long water waves‡ in an infinite two-dimensional sheet of water of depth $h$ and density $\rho$.*

The motion is supposed to be initiated from rest at $t = 0$ by a variable pressure $P(r,t)$ acting on the surface (for simplicity we consider only the case of symmetry about the origin).

Then, writing $F(r,t) = \dfrac{1}{\rho c^2}\dfrac{\partial P}{\partial t}$, and $c = \sqrt{(gh)}$, we have to solve

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -F(r,t),$$

with, at $t = 0$, $\phi$ and $\partial\phi/\partial t$ zero for all $r$.

The subsidiary equation is

$$\frac{d^2\bar{\phi}}{dr^2}+\frac{1}{r}\frac{d\bar{\phi}}{dr}-\frac{p^2}{c^2}\bar{\phi} = -\bar{F}(r,p). \tag{1}$$

† Multiply both sides of (4) by $e^{-im'x}$ and integrate with respect to $x$ from $-\infty$ to $\infty$. Then we obtain

$$\int\limits_{-\infty}^{\infty} e^{-im'x}\,dx\int\limits_{-\infty}^{\infty}(p^2+|m|g)\psi(m)e^{imx}\,dm = g\int\limits_{-\infty}^{\infty} e^{-im'x}f(x)\,dx.$$

By § 30 (2) the left-hand side equals

$$2\pi(p^2+|m'|g)\psi(m'),$$

and the result (5) follows on writing $m$ for $m'$ and $x'$ for $x$.

‡ Lamb, loc. cit., § 195 et seq.

To solve this we seek the Green's function,†

$$G(r,\xi) = AI_0\left(\frac{pr}{c}\right), \quad 0 \leqslant r < \xi,$$

$$= BK_0\left(\frac{pr}{c}\right), \quad r > \xi,$$

where $$[G]_{\xi-0}^{\xi+0} = BK_0\left(\frac{p\xi}{c}\right) - AI_0\left(\frac{p\xi}{c}\right) = 0$$

and $$\left[r\frac{dG}{dr}\right]_{\xi-0}^{\xi+0} = B\frac{p\xi}{c}K_0'\left(\frac{p\xi}{c}\right) - A\frac{p\xi}{c}I_0'\left(\frac{p\xi}{c}\right) = 1.$$

Solving‡ $A = -K_0(p\xi/c)$, $B = -I_0(p\xi/c)$, and the solution of (1) becomes

$$\bar{\phi}(r,p) = K_0\left(\frac{pr}{c}\right)\int_0^r I_0\left(\frac{p\eta}{c}\right)\eta\bar{F}(\eta,p)\,d\eta + I_0\left(\frac{pr}{c}\right)\int_r^\infty K_0\left(\frac{p\eta}{c}\right)\eta\bar{F}(\eta,p)\,d\eta. \tag{2}$$

Now for simplicity suppose the applied surface pressure concentrated in a vanishingly small circle about the origin and let

$$\int_0^\infty rF(r,t)\,dr = g(t),$$

so that $$\int_0^\infty r\bar{F}(r,p)\,dr = \bar{g}(p).$$

Then, since $I_0(0) = 1$ and $\bar{F}$ vanishes except near the origin, (2) may be replaced by

$$\bar{\phi}(r,p) = K_0\left(\frac{pr}{c}\right)\int_0^\infty \eta\bar{F}(\eta,p)\,d\eta$$

$$= \bar{g}(p)K_0\left(\frac{pr}{c}\right). \tag{3}$$

To determine $\phi$ from this we require first the function whose transform is $K_0(pr/c)$. By the Inversion Theorem this is

$$I = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}K_0\left(\frac{\lambda r}{c}\right)\,d\lambda. \tag{4}$$

† See § 68. The result (2) may also be obtained by variation of parameters.

‡ $I_0(z)K_0'(z) - K_0(z)I_0'(z) = -1/z$.

(i) If $t < r/c$ we use the contour of Fig. 17. $AB$ is distant $\gamma$ from the imaginary axis and parallel to it, the circle $C$ is of radius $R$ which will tend to infinity.

The integrand of (4) has no pole or branch-point inside or on this contour. Using the asymptotic expansion for the Bessel function, it may be shown that when $t < r/c$ the integral over $C$ vanishes as $R \to \infty$. Thus, from Cauchy's theorem,

$$I = 0, \quad \text{when } t < \frac{r}{c}.$$

(ii) If $t > r/c$, we use the path of Fig. 11, since $K_0(\lambda r/c)$ has a branch-point at the origin. It is easily verified that the integrals over the large and small circles approach zero as their radii approach $\infty$ and $0$ respectively. The integrand has no poles within the contour, so $I$ reduces to a sum of integrals over $CD$ and $EF$. Putting $\lambda = \rho e^{-i\pi}$ and $\lambda = \rho e^{i\pi}$ on these we obtain

$$I = \frac{1}{2\pi i}\int\limits_0^\infty e^{-\rho t}\left[K_0\left(\frac{r\rho}{c}e^{-i\pi}\right) - K_0\left(\frac{r\rho}{c}e^{i\pi}\right)\right]d\rho = \int\limits_0^\infty e^{-\rho t}J_0\left(\frac{ir\rho}{c}\right)d\rho$$

$$= \left[t^2 - \frac{r^2}{c^2}\right]^{-\frac{1}{2}}, \quad \text{when } t > \frac{r}{c},$$

where, in the last line, we have used the result† Appendix II (35). Thus, finally,‡

$$\left.\begin{aligned} I &= 0, \quad t < \frac{r}{c}, \\ &= \left[t^2 - \frac{r^2}{c^2}\right]^{-\frac{1}{2}}, \quad t > \frac{r}{c}. \end{aligned}\right\} \qquad (5)$$

So from (3) and (5), using § 3, Theorem VI,

$$\phi(r,t) = \int\limits_{r/c}^{t} g(t-\tau)\frac{d\tau}{(\tau^2 - r^2/c^2)^{\frac{1}{2}}}$$

$$= \int\limits_0^{\cosh^{-1}ct/r} g\left(t - \frac{r}{c}\cosh u\right)du.$$

† Putting $n = 0$, and remembering $I_0(iz) = J_0(z)$.

‡ This result may also be obtained from the integral (*W.B.F.*, p. 181 (5))

$$K_0(z) = \int\limits_0^\infty e^{-z\cosh u}\,du.$$

**82.** *Wave motion in air under gravity*† *and at constant temperature.*

Let the $x$-axis be taken vertically upwards and let $\xi$ be the vertical displacement of the particles initially at $x$. Suppose the motion initiated from rest at $t = 0$ by a small displacement $\xi_0$ of the plane $x = 0$.

We have to solve

$$\frac{\partial^2\xi}{\partial x^2}-\frac{2k}{c}\frac{\partial\xi}{\partial x}-\frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2} = 0, \quad t > 0, x > 0,$$

with $\xi = \xi_0$, when $x = 0$, $t > 0$, where $k = \gamma g/2c$, $\gamma$ the ratio of specific heats.

The subsidiary equation is

$$\frac{d^2\bar{\xi}}{dx^2}-\frac{2k}{c}\frac{d\bar{\xi}}{dx}-\frac{p^2}{c^2}\bar{\xi} = 0,$$

with $\bar{\xi} = \xi_0/p$, when $x = 0$.

Choosing the solution which remains finite as $x \to \infty$, we have

$$\bar{\xi} = \frac{\xi_0}{p}e^{kx/c-(x/c)\sqrt{(p^2+k^2)}}.$$

Thus, using the Inversion Theorem,

$$\xi = \frac{\xi_0 e^{kx/c}}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t-(x/c)\sqrt{(\lambda^2+k^2)}}\,d\lambda}{\lambda}.$$

Integrals of this type‡ will be discussed in detail in Chapter IX, §§ 90–2.

Assuming the result § 90 (16) and replacing $v$ by $c$, $\alpha$ by $\frac{1}{2}ik$, $\beta$ by $-\frac{1}{2}ik$, $\sigma = \alpha-\beta$ by $ik$, and $\rho = \alpha+\beta$ by 0, we have

$$\xi = 0, \quad t < x/c,$$

$$= \xi_0 e^{kx/c}\left\{1-\frac{kx}{c}\int\limits_{x/c}^{t}\frac{J_1[k\sqrt{(\tau^2-x^2/c^2)}]}{[\tau^2-x^2/c^2]^{\frac{1}{2}}}\,d\tau\right\}, \; t > x/c.$$

† Lamb, loc. cit., § 309.

‡ This problem has been given since the result can be derived from § 90. The integrals arising in problems of physical interest may be evaluated by the type of manipulation used in §§ 90–2.

# CHAPTER IX

## ELECTRIC TRANSMISSION LINES

**83.** Let $R$, $L$, $C$, $G$ be the resistance, inductance, capacity, and leakage conductance per unit length of the line.† Let $V$ be the potential, and $I$ the current, at the point $x$ of the line at time $t$, and let $V^{(0)}$ and $I^{(0)}$ be their values at $t = 0$.

$V$ and $I$ have to satisfy the differential equations

$$\left.\begin{aligned} L\frac{\partial I}{\partial t}+RI &= -\frac{\partial V}{\partial x}, \\ C\frac{\partial V}{\partial t}+GV &= -\frac{\partial I}{\partial x}. \end{aligned}\right\} \quad t > 0. \tag{1}$$

Multiplying these by $e^{-pt}$, $p > 0$, and integrating with respect to $t$ from 0 to $\infty$, we obtain the subsidiary equations

$$\begin{aligned} (Lp+R)\bar{I} &= -\frac{d\bar{V}}{dx}+LI^{(0)}, \\ (Cp+G)\bar{V} &= -\frac{d\bar{I}}{dx}+CV^{(0)}. \end{aligned} \tag{2}$$

Eliminating $\bar{I}$ we have the ordinary differential equation for $\bar{V}$,

$$\frac{d^2\bar{V}}{dx^2}-(Lp+R)(Cp+G)\bar{V} = L\frac{dI^{(0)}}{dx}-C(Lp+R)V^{(0)}, \tag{3}$$

and $\bar{I}$ is given by

$$\bar{I} = -\frac{1}{Lp+R}\frac{d\bar{V}}{dx}+\frac{LI^{(0)}}{Lp+R}. \tag{4}$$

If we write $$q^2 = (Lp+R)(Cp+G), \tag{5}$$

the complementary function of (3) is $Ae^{qx}+Be^{-qx}$, where $A$ and $B$ are to be found from the terminal conditions.

The general case in which none of $R$, $L$, $G$, $C$ vanish is relatively difficult, but there are various special cases in which

† These are taken to be constant. The case of a 'non-uniform' line, i.e. one in which they are functions of $x$, is considered briefly in § 94.

$q$ takes a simple form and the problems are of the types treated in Chapters V, VI, and VII:

(i) $R = G = 0, \quad q = p\sqrt{(LC)}$;

(ii) $\dfrac{R}{L} = \dfrac{G}{C}, \qquad q = \sqrt{(LC)}\left(p+\dfrac{R}{L}\right)$;

(iii) $L = G = 0, \quad q = \sqrt{(RC)}\sqrt{p}$;

(iv) $L = 0, \qquad q = \sqrt{(RC)}\sqrt{(p+G/C)}$.

Of these, (ii), Heaviside's 'distortionless line', and (iii), which approximates to slow signalling on a submarine cable, are of some importance. In (i) and (ii) the solutions are of wave type, while in (iii) and (iv) they are of diffusive type, as in the linear flow of heat.

**84.** *Problems in which $L = G = 0$ already solved as problems in the linear flow of heat.*

Ex. 1. *Semi-infinite line $x > 0$. Initial current and potential zero. At $t = 0$ an alternating E.M.F. $a \cos \omega t$ connected at $x = 0$.*

We have to solve § 83 (1) with $L = G = 0$, and terminal conditions

$$\left.\begin{aligned} V &= a\cos\omega t, \quad \text{when } x = 0, \\ V &\text{ finite}, \quad \text{when } x \to \infty. \end{aligned}\right\} \quad t > 0.$$

Thus, by § 83 (3), the subsidiary equation is

$$\frac{d^2\bar{V}}{dx^2} - RCp = 0, \quad x > 0,$$

to be solved with

$$\bar{V} = \frac{ap}{p^2+\omega^2}, \quad \text{when } x = 0,$$

and
$$\bar{V} \text{ finite} \quad \text{as } x \to \infty.$$

This is exactly the problem of § 47 with $RC$ in place of $1/\kappa$, and so, making this change in § 47 (9), we have the solution

$$V = ae^{-x\sqrt{(\frac{1}{2}RC\omega)}}\cos\{\omega t - x\sqrt{(\tfrac{1}{2}RC\omega)}\} - \frac{a}{\pi}\int\limits_0^\infty e^{-\rho t}\sin x\sqrt{(RC\rho)}\frac{\rho\, d\rho}{\rho^2+\omega^2}.$$

Ex. 2. *Semi-infinite line $x > 0$. Initial current and potential zero. At $t = 0$ a constant E.M.F. $V_0$ applied at $x = 0$.*

Replacing $\kappa$ by $1/RC$ in § 39 (8), we obtain

$$V = V_0\left[1-\operatorname{erf}\tfrac{1}{2}x\left(\frac{RC}{t}\right)^{\frac{1}{2}}\right].$$

Ex. 3. *Line of length $l$ with initial potential $V_0$ and zero initial current. The end $x = l$ kept at potential $V_1$ for $t > 0$. Open circuit at $x = 0$ for $t > 0$.*

Writing $1/RC$ for $\kappa$ in the result of § 40, we get

$$V = V_1+\frac{4(V_1-V_0)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(n-\frac{1}{2})^2\pi^2t/(l^2RC)}\cos\frac{(2n-1)\pi x}{2l}.$$

Ex. 4. *Line of length $l$ with zero initial current. The initial potential of the line an arbitrary function $f(x)$. The ends $x = 0$ and $x = l$ kept at zero potential, $t > 0$.*

Putting $L = G = 0$, $I^{(0)} = 0$, $V^{(0)} = f(x)$ in § 83 (3), the subsidiary equation is

$$\frac{d^2\bar{V}}{dx^2}-RCp = -RCf(x), \quad 0 < x < l,$$

to be solved with $\bar{V} = 0$, when $x = 0$ and $x = l$.

The solution of the corresponding problem in conduction of heat has been given in § 50 (p. 119, small print). Thus, writing $1/RC$ for $\kappa$, we get

$$V = \frac{2}{l}\sum_{n=1}^{\infty} e^{-n^2\pi^2t/RCl^2}\sin\frac{n\pi}{l}x\int\limits_0^l f(\xi)\sin\frac{n\pi}{l}\xi\,d\xi.$$

**85.** *Line of length $l$, $L = G = 0$, earthed at $x = l$. Initial current and potential zero. At $t = 0$ an alternating E.M.F. $e^{i\omega t}$ is applied at the end $x = 0$.*

We have to solve § 83 (1) with $L = G = 0$ and

$$V = 0, \quad \text{when } x = l,\ t > 0,$$
$$V = e^{i\omega t}, \quad \text{when } x = 0,\ t > 0,$$
$$V^{(0)} = I^{(0)} = 0, \quad 0 < x < l.$$

From § 83 (3) the subsidiary equation is

$$\frac{d^2\bar{V}}{dx^2}-RCp\bar{V} = 0,$$

to be solved with

$$\bar{V} = \frac{1}{p-i\omega} \qquad \text{when } x = 0,$$

and $$\bar{V} = 0, \qquad \text{when } x = l.$$

The solution is

$$\bar{V} = \frac{1}{p-i\omega}\cdot\frac{\sinh(l-x)\sqrt{(RCp)}}{\sinh l\sqrt{(RCp)}}.$$

Therefore, using the Inversion Theorem,

$$V = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda-i\omega}\,\frac{\sinh(l-x)\sqrt{(RC\lambda)}}{\sinh l\sqrt{(RC\lambda)}}\,d\lambda.$$

The integrand is a single-valued† function of $\lambda$ with simple poles at $i\omega$, and $-(n^2\pi^2/l^2RC)$, $n = 1, 2,\ldots$.

We take the contour $ABCA$ of Fig. 10 and choose its radius

$$R = (n+\tfrac{1}{2})^2\pi^2/(l^2RC),$$

so that $\Gamma$ does not pass through any pole of the integrand. The integral over $\Gamma$ tends to zero as $n \to \infty$. Thus $\int\limits_{\gamma-i\infty}^{\gamma+i\infty}$ may be replaced by the limit of the integral over $ABCA$ as $n \to \infty$, and by Cauchy's theorem this equals $2\pi i$ times the sum of the residues of the integrand at poles within the contour.

The pole at $i\omega$ has residue

$$e^{i\omega t}\frac{\sinh(l-x)\sqrt{(RC\omega i)}}{\sinh l\sqrt{(RC\omega i)}} = e^{i\omega t}\frac{\sinh(l-x)(1+i)\sqrt{(\frac{1}{2}RC\omega)}}{\sinh l(1+i)\sqrt{(\frac{1}{2}RC\omega)}}$$

and the pole at $-n^2\pi^2/l^2RC$ has residue‡

$$\frac{2}{\pi}\,\frac{(-1)^n e^{-n^2\pi^2t/(l^2RC)}}{n\{1+i(RCl^2\omega/n^2\pi^2)\}}\sin\frac{n\pi(l-x)}{l}.$$

† Since, using the series for $\sinh x$,

$$\frac{\sinh(l-x)\sqrt{(RC\lambda)}}{\sinh l\sqrt{(RC\lambda)}} = \frac{l-x}{l}\,\frac{1+\frac{1}{6}(l-x)^2RC\lambda+\ldots}{1+\frac{1}{6}l^2RC\lambda+\ldots}$$

involves only integral powers of $\lambda$.

‡ $$\left[\frac{d}{d\lambda}\sinh l\sqrt{(RC\lambda)}\right]_{\lambda=-n^2\pi^2/l^2RC} = \left[\frac{l}{2}\sqrt{\left(\frac{RC}{\lambda}\right)}\cosh l\sqrt{(RC\lambda)}\right]_{\lambda=-n^2\pi^2/l^2RC} = \frac{RCl^2}{2in\pi}\cos n\pi.$$

Therefore

$$V = e^{i\omega t}\frac{\sinh(l-x)(1+i)\sqrt{(\frac{1}{2}RC\omega)}}{\sinh l(1+i)\sqrt{(\frac{1}{2}RC\omega)}}+$$
$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-n^2\pi^2 t/(l^2RC)}}{n\{1+i(RCl^2\omega/n^2\pi^2)\}}\sin\frac{n\pi(l-x)}{l}.$$

If the applied E.M.F. is $\cos\omega t$, we have to take the real part of this, namely,

$$\left\{\frac{\cosh(l-x)\sqrt{(2RC\omega)}-\cos(l-x)\sqrt{(2RC\omega)}}{\cosh l\sqrt{(2RC\omega)}-\cos l\sqrt{(2RC\omega)}}\right\}^{\frac{1}{2}}\cos(\omega t+\phi-\phi')+$$
$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n\{1+R^2C^2\omega^2l^4/n^4\pi^4\}^{\frac{1}{2}}}e^{-n^2\pi^2 t/l^2RC}\cos\theta_n\sin\frac{n\pi(l-x)}{l},$$

where

$$\tan\phi = \tan(l-x)\sqrt{(\tfrac{1}{2}RC\omega)}\coth(l-x)\sqrt{(\tfrac{1}{2}RC\omega)},$$
$$\tan\phi' = \tan l\sqrt{(\tfrac{1}{2}RC\omega)}\coth l\sqrt{(\tfrac{1}{2}RC\omega)},$$
$$\tan\theta_n = RCl^2\omega/n^2\pi^2.$$

**86.** *Line of length $l$. At $x = l$ the line is earthed through an impedance $z_2$. At $x = 0$ a constant E.M.F. is applied at $t = 0$ through an impedance $z_1$. Initial charge and current zero.*

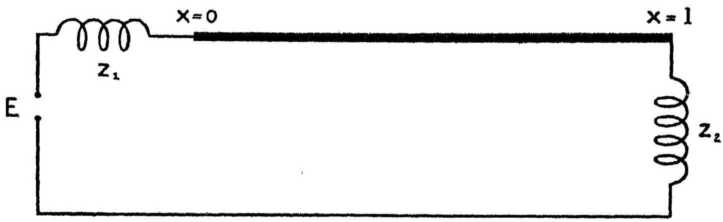


FIG. 16.

Let $V_0$, $I_0$, $V_l$, $I_l$ be the potentials and currents at $x = 0$ and $x = l$ respectively. Then for the concentrated impedances the subsidiary equations are

$$z_1\bar{I}_0 = \frac{E}{p}-\bar{V}_0, \qquad (1)$$
$$z_2\bar{I}_l = \bar{V}_l.$$

For the line itself, by § 83 (3), the subsidiary equation is

$$\frac{d^2\bar{V}}{dx^2}-q^2\bar{V}=0, \quad 0<x<l,$$

where $q$ is given by § 83 (5).

The solution of this is

$$\bar{V}=A\sinh qx+B\cosh qx, \tag{2}$$

and, by § 83 (4),

$$\bar{I}=-\frac{q}{Lp+R}[A\cosh qx+B\sinh qx]. \tag{3}$$

$A$ and $B$ are to be found by substituting in the terminal conditions (1) which give

$$-\frac{q}{Lp+R}z_1 A=\frac{E}{p}-B,$$

$$-\frac{q}{Lp+R}[A\cosh ql+B\sinh ql]z_2=A\sinh ql+B\cosh ql.$$

Solving for $A$ and $B$ and substituting in (2) we obtain

$$\bar{V}=\frac{E}{p}\frac{qz_2\cosh q(l-x)+(R+Lp)\sinh q(l-x)}{q(z_1+z_2)\cosh ql+[(R+Lp)+z_1z_2(G+Cp)]\sinh ql}. \tag{4}$$

As a simple example let $L=G=0$, $z_1=0$, $z_2=1/C_2p$; then (4) becomes

$$\bar{V}=\frac{E}{p}\frac{\sinh(l-x)\sqrt{(RCp)}+\sqrt{(C/RC_2^2p)}\cosh(l-x)\sqrt{(RCp)}}{\sinh l\sqrt{(RCp)}+\sqrt{(C/RC_2^2p)}\cosh l\sqrt{(RCp)}}.$$

Therefore, using the Inversion Theorem,

$$V=\frac{E}{2\pi i}\times$$

$$\times\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda}\frac{\sinh(l-x)\sqrt{(RC\lambda)}+\sqrt{(C/RC_2^2\lambda)}\cosh(l-x)\sqrt{(RC\lambda)}}{\sinh l\sqrt{(RC\lambda)}+\sqrt{(C/RC_2^2\lambda)}\cosh l\sqrt{(RC\lambda)}}\,d\lambda. \tag{5}$$

The integrand is a single-valued function of $\lambda$ with simple poles at $\lambda=0$, and $\lambda=-(\alpha_s^2/RCl^2)$, $s=1, 2,\ldots$, where $\pm\alpha_1$, $\pm\alpha_2,\ldots$ are the roots of

$$\alpha\tan\alpha=\frac{lC}{C_2}.$$

It is easily verified that these are all real and simple.

We use the contour of Fig. 10 with radius $(n+\frac{1}{2})^2\pi^2/(RCl^2)$. Then $\Gamma$ does not pass through any pole of the integrand. The integral over $\Gamma$ tends to zero as $n \to \infty$. Thus, by Cauchy's theorem, the line integral in (5) may be replaced by $2\pi i$ times the sum of the residues at its poles.

The residue at $\lambda = 0$ is 1.

The residue at $\lambda = -(\alpha_s^2/RCl^2)$ is†

$$2lCe^{-\alpha_s^2 t/(RCl^2)}\frac{C_2\,\alpha_s \sin\alpha_s(l-x)/l - lC\cos\alpha_s(l-x)/l}{(lCC_2+l^2C^2+C_2^2\,\alpha_s^2)\alpha_s\sin\alpha_s}.$$

Therefore

$$V = E+2lCE\sum_{s=1}^{\infty} e^{-\alpha_s^2 t/(RCl^2)}\frac{C_2\,\alpha_s \sin\alpha_s(l-x)/l - lC\cos\alpha_s(l-x)/l}{(lCC_2+l^2C^2+C_2^2\,\alpha_s^2)\alpha_s\sin\alpha_s}.$$

**87.** *Line of length $l$. Initial current and potential zero. Open circuit at $x = l$. A constant E.M.F. $E$ applied at $t = 0$ at the end $x = 0$.*

We have to solve § 83 (1) with

$$V = E, \quad \text{when } x = 0,\ t > 0,$$

$$I = 0, \quad \text{when } x = l,\ t > 0,$$

$$V_0^{(0)} = I^{(0)} = 0, \quad 0 < x < l.$$

The subsidiary equation § 83 (3) is

$$\frac{d^2\bar{V}}{dx^2} - q^2\bar{V} = 0,$$

where $q^2 = (Lp+R)(Cp+G)$.

This has to be solved with

$$\bar{V} = \frac{E}{p}, \quad \text{when } x = 0,$$

and, by § 83 (4),

$$\frac{d\bar{V}}{dx} = 0, \quad \text{when } x = l.$$

† $$\left[\lambda\frac{d}{d\lambda}\left\{\sinh l\sqrt{(RC\lambda)}+\sqrt{\left(\frac{C}{RC_2^2\,\lambda}\right)}\cosh l\sqrt{(RC\lambda)}\right\}\right]_{\lambda=-\alpha^2/RCl^2} = i\sin\alpha_s\left\{\frac{1}{2}+\frac{lC}{2C_2}+\frac{C_2\,\alpha_s^2}{2lC}\right\}.$$

The solution of this is

$$\bar{V} = \frac{E}{p}\,\frac{\cosh q(l-x)}{\cosh ql}. \tag{1}$$

Therefore, by the Inversion Theorem,

$$V = \frac{E}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda}\,\frac{\cosh \mu(l-x)}{\cosh \mu l}\,d\lambda, \tag{2}$$

where $\mu = [(L\lambda+R)(C\lambda+G)]^{\frac{1}{2}}$.

The integrand of (2) is a single-valued† function of $\lambda$ with poles at $\lambda = 0$, and at the roots of $\cosh \mu l = 0$, i.e. at the roots of

$$(L\lambda+R)(C\lambda+G) = -\frac{(2n+1)^2\pi^2}{4l^2}, \quad n = 0, 1, 2,\dots. \tag{3}$$

Using the notation

$$\rho = \frac{R}{2L}+\frac{G}{2C}, \qquad \sigma = \frac{R}{2L}-\frac{G}{2C}, \qquad v = (LC)^{-\frac{1}{2}}. \tag{4}$$

(3) becomes

$$\lambda^2+2\rho\lambda+\rho^2-\sigma^2+\frac{(2n+1)^2\pi^2v^2}{4l^2} = 0, \quad n = 0, 1, 2,\dots,$$

the roots of which are $\qquad -\rho\pm i\nu_n, \qquad$ (5)

where $\qquad \nu_n = \left\{\frac{(2n+1)^2\pi^2v^2}{4l^2}-\sigma^2\right\}^{\frac{1}{2}}. \qquad$ (6)

We shall assume that $\nu_n$ is real for all $n$, i.e. $\frac{\pi v}{l} > \frac{R}{L}-\frac{G}{C}$ so that all the roots (5) are complex with real part $-\rho$; if this is not the case, the roots for small $n$ will be real and negative and the form of the solution slightly different.

Using the contour of Fig. 10, the usual argument shows that the line integral in (2) may be replaced by $2\pi i$ times the sum of the residues of the integrand at its poles.

The residue at $\lambda = 0$ is

$$\frac{\cosh(l-x)\sqrt{(RG)}}{\cosh l\sqrt{(RG)}}.$$

† Since $\quad \frac{\cosh \mu(l-x)}{\cosh \mu l} = \frac{1+\frac{1}{2}(l-x)^2(L\lambda+R)(C\lambda+G)+\dots}{1+\frac{1}{2}l^2(L\lambda+R)(C\lambda+G)+\dots},$
and thus contains only integral powers of $\lambda$.

Also, since

$$\left[\lambda\frac{d}{d\lambda}\cosh\mu l\right]_{\lambda=-\rho+i\nu_n} = \left[\frac{l\lambda(\lambda+\rho)}{\mu v^2}\sinh\mu l\right]_{\lambda=-\rho+i\nu_n}$$
$$= \frac{2l^2}{(2n+1)\pi v^2}(-1)^{n+1}\nu_n(\nu_n+i\rho), \tag{7}$$

the residue at $\lambda = -\rho+i\nu_n$ is

$$(-1)^{n+1}\frac{(2n+1)\pi v^2}{2l^2\nu_n(\nu_n^2+\rho^2)^{\frac{1}{2}}}e^{-\rho t+i\nu_n t-i\theta_n}\cos(l-x)(2n+1)\pi/(2l),$$

where $\tan\theta_n = \rho/\nu_n$.

Therefore, finally,

$$V = E\frac{\cosh(l-x)\sqrt{(RG)}}{\cosh l\sqrt{(RG)}} - \tag{8}$$
$$-\frac{\pi v^2 E}{l^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)}{\nu_n(\nu_n^2+\rho^2)^{\frac{1}{2}}}e^{-\rho t}\cos(\nu_n t-\theta_n)\cos\frac{(l-x)(2n+1)\pi}{2l}.$$

**88.** The solutions of the problems so far considered have all been obtained as trigonometrical series. In discussing mechanical vibration problems in Chapter V, § 44, an alternative method of solution was developed in which the hyperbolic functions in the transform were expanded in a series of exponentials; this procedure gave a solution with a convenient physical interpretation in terms of successively reflected waves. The same method can be applied to transmission-line problems.

As a first example consider the problem of § 87 with

$$R = G = 0.$$

Then $q^2 = LCp^2 = p^2/v^2$, in the notation of § 87 (4), and putting this value of $q$ in § 87 (1) we have

$$\bar{V} = \frac{E}{p}\frac{\cosh p(l-x)/v}{\cosh pl/v}$$
$$= \frac{E}{p}\frac{e^{-px/v}[1+e^{-2p(l-x)/v}]}{1+e^{-2pl/v}}$$
$$= \frac{E}{p}e^{-px/v}[1+e^{-2p(l-x)/v}][1-e^{-2pl/v}+e^{-4pl/v}-\ldots]$$
$$= \frac{E}{p}\left\{\begin{matrix} e^{-px/v}-e^{-p(x+2l)/v}+e^{-p(x+4l)/v}-\ldots+ \\ +e^{-p(2l-x)/v}-e^{-p(4l-x)/v}+e^{-p(6l-x)/v}-\ldots \end{matrix}\right\}. \tag{1}$$

It follows from Theorem V, § 3, that the function whose transform is $(E/p)e^{-ap}$ is

$$\left.\begin{array}{ll} 0, & \text{for } t < a, \\ E, & \text{for } t > a, \end{array}\right\}$$

which for shortness we shall write $EH(t-a)$, where $H(t)$ is defined by

$$\left.\begin{aligned} H(t) &= 0, \quad t < 0, \\ &= 1, \quad t > 0. \end{aligned}\right\} \tag{2}$$

Applying this to the terms of (1) successively we find

$$\begin{aligned} V &= 0, && \text{if} \quad 0 < t < x/v, \\ &= E, && \text{if} \quad x/v < t < (2l-x)/v, \\ &= 2E, && \text{if} \quad (2l-x)/v < t < (2l+x)/v, \\ &= E, && \text{if} \quad (2l+x)/v < t < (4l-x)/v, \\ &= 0, && \text{if} \quad (4l-x)/v < t < (4l+x)/v, \end{aligned}$$

etc.

Or in terms of the function $H(t)$

$$\frac{V}{E} = H\left(t-\frac{x}{v}\right)+H\left(t-\frac{2l-x}{v}\right)-H\left(t-\frac{2l+x}{v}\right)-H\left(t-\frac{4l-x}{v}\right)+\dots. \tag{3}$$

Thus, if we regard the potential as propagated with velocity $v$, the potential at $x$ is zero till the direct wave reaches it, $E$ from this time till the wave reflected from $x = l$ reaches it, then $2E$ till the twice reflected wave reaches it, and so on.

If we form the Fourier series for the 'step function' (3) we obtain the trigonometrical series § 87 (8) for the case $R = G = 0$.

Treating in the same way the problem of § 87 with none of $R$, $L$, $G$, $C$ zero we have from § 87 (1)

$$\bar{V} = \frac{E}{p}\frac{\cosh q(l-x)}{\cosh ql} = \frac{E}{p}[e^{-qx}+e^{-q(2l-x)}-e^{-q(2l+x)}-e^{-q(4l-x)}+\dots], \tag{4}$$

where now

$$q = [(Lp+R)(Cp+G)]^{\frac{1}{2}} = \frac{1}{v}[(p+\rho)^2-\sigma^2]^{\frac{1}{2}}$$

in the notation of § 87 (4).

$V$ can be obtained from this when the function whose transform is $(1/p)e^{-qx}$ is known; this will be found in § 90, and the solution completed in § 93.

In the case of the 'distortionless' line in which $\sigma = 0$, $q = (p+\rho)/v$, (4) becomes

$$\bar{V} = \frac{E}{p}[e^{-x(p+\rho)/v}+e^{-(2l-x)(p+\rho)/v}-e^{-(2l+x)(p+\rho)/v}-\dots].$$

Thus

$$\frac{V}{E} = e^{-\rho x/v}H\left(t-\frac{x}{v}\right)+e^{-\rho(2l-x)/v}H\left(t-\frac{2l-x}{v}\right)-$$
$$-e^{-\rho(2l+x)/v}H\left(t-\frac{2l+x}{v}\right)-\dots.$$

The structure of this is the same as that of (3) except for the attenuation factors involving the distance which each wave has travelled.

Finally, consider a *distortionless line of length $l$, with open circuit at $x = l$, and zero initial current and potential. An E.M.F. $f(t)$ is applied at $x = 0$, $t > 0$.*

We have only to replace $E/p$ in (4) by $\bar{f} = \int\limits_0^\infty e^{-pt}f(t)\,dt$, and we obtain

$$\bar{V} = \bar{f}\frac{\cosh q(l-x)}{\cosh ql} = \bar{f}[e^{-qx}+e^{-q(2l-x)}-e^{-q(2l+x)}-e^{-q(4l-x)}+\dots], \quad (5)$$

where $q = (p+\rho)/v$.

To find $V$ from (5) we require the function whose transform is $e^{-ap}\bar{f}$, where $a$ is a constant. By § 3, Theorem V, this is $f(t-a)H(t-a)$. Using this result in (5), we find

$$V = e^{-\rho x/v}f\left(t-\frac{x}{v}\right)H\left(t-\frac{x}{v}\right)+$$
$$+e^{-\rho(2l-x)/v}f\left(t-\frac{2l-x}{v}\right)H\left(t-\frac{2l-x}{v}\right)-\dots.$$

**89.** *A line of length $l$ with initial potential $V^{(0)} = f(x)$ and zero initial current. At $t = 0$ the end $x = 0$ is earthed, the end $x = l$ being left insulated.*

By § 83 (3) the subsidiary equation is

$$\frac{d^2\bar{V}}{dx^2}-q^2\bar{V} = -C(Lp+R)f(x), \tag{1}$$

where $q^2 = (Lp+R)(Cp+G)$.

This has to be solved with

$$\bar{V} = 0, \quad \text{at } x = 0,$$

and
$$\frac{d\bar{V}}{dx} = 0, \quad \text{at } x = l. \tag{2}$$

By § 3, Theorem VI, a Particular Integral of (1) is

$$\frac{C(Lp+R)}{q} \int^{x} f(\xi)\sinh q(x-\xi)\, d\xi,$$

and thus the general solution of (1) is

$$\bar{V} = A\sinh qx+B\cosh qx-\frac{C(Lp+R)}{q}\int_0^x f(\xi)\sinh q(x-\xi)\, d\xi,$$

where $A$ and $B$ are to be found by substituting in the terminal conditions (2). These give

$$B = 0$$

and
$$qA\cosh ql-C(Lp+R)\int_0^l f(\xi)\cosh q(l-\xi)\, d\xi = 0.$$

Therefore†

$$\bar{V} = \frac{C(Lp+R)}{q}\frac{\cosh q(l-x)}{\cosh ql}\int_0^x f(\xi)\sinh q\xi\, d\xi +$$

$$+\frac{C(Lp+R)}{q}\frac{\sinh qx}{\cosh ql}\int_x^l f(\xi)\cosh q(l-\xi)\, d\xi. \tag{3}$$

*As an example, suppose*

$$\left.\begin{aligned} f(x) &= 0, & 0 < x < a,\\ &= Q/C, & a < x < b,\\ &= 0, & b < x < l. \end{aligned}\right\}$$

† This could also have been obtained by using the Green's function as in § 68, or by using variation of parameters as in § 42.

We determine the potential for $x < a$. In this case (3) becomes

$$\bar{V} = \frac{Q(Lp+R)}{q^2}\frac{\sinh qx}{\cosh ql}[\sinh q(l-a)-\sinh q(l-b)].$$

Therefore, by the Inversion Theorem,

$$V = \frac{Q}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{C\lambda+G}\frac{\sinh \mu x[\sinh \mu(l-a)-\sinh \mu(l-b)]\,d\lambda}{\cosh \mu l}, \tag{4}$$

where $\mu = [(L\lambda+R)(C\lambda+G)]^{\frac{1}{2}}$.

The integrand is a single-valued function of $\lambda$ with poles at the roots of

$$\cosh \mu l = 0.$$

These have been found in § 87 (3) and (5) to be

$$-\rho \pm i\nu_n, \quad n = 0, 1, 2, \ldots,$$

where, as in § 87, we assume $\nu_n$ real for all $n$.

As before, using Fig. 10, the line integral in (4) may be replaced by $2\pi i$ times the sum of the residues at the poles of its integrand.

By § 87 (7),

$$\left[\frac{d}{d\lambda}\cosh \mu l\right]_{\lambda=-\rho+i\nu_n} = \frac{2il^2(-1)^n\nu_n}{(2n+1)\pi v^2},$$

and thus the residue at $\lambda = -\rho+i\nu_n$ is, writing $\tan\theta_n = \sigma/\nu_n$,

$$\frac{(2n+1)\pi v^2}{2Cl^2\nu_n(\nu_n^2+\sigma^2)^{\frac{1}{2}}}e^{-\rho t+i\nu_n t-i\theta_n}\sin\frac{(2n+1)\pi x}{2l}\times$$
$$\times\left\{\cos\frac{(2n+1)\pi a}{2l}-\cos\frac{(2n+1)\pi b}{2l}\right\}.$$

Therefore

$$V = \frac{Q\pi v^2}{Cl^2}e^{-\rho t}\sum_{n=0}^{\infty}\frac{(2n+1)}{\nu_n(\nu_n^2+\sigma^2)^{\frac{1}{2}}}\cos(\nu_n t-\theta_n)\sin\frac{(2n+1)\pi x}{2l}\times$$
$$\times\left\{\cos\frac{(2n+1)\pi a}{2l}-\cos\frac{(2n+1)\pi b}{2l}\right\},$$

for $x < a$. The other ranges of $x$ may be discussed similarly.

**90.** *Uniform semi-infinite transmission line, $x > 0$, with zero initial current and potential. At $t = 0$ a constant E.M.F., $E$, applied at the end $x = 0$.*

From § 83 (3) the subsidiary equation is

$$\frac{d^2\bar{V}}{dx^2} - q^2\bar{V} = 0, \tag{1}$$

where $$q^2 = (Lp+R)(Cp+G) = \frac{1}{v^2}(p+2\alpha)(p+2\beta), \tag{2}$$

with the notation of § 87 (4), namely,

$$v = (LC)^{-\frac{1}{2}}, \qquad \alpha = R/(2L), \qquad \beta = G/(2C),$$
$$\rho = \alpha+\beta, \qquad \sigma = \alpha-\beta. \tag{3}$$

The solution of (1), which is finite as $x \to \infty$ and takes the value $E/p$ for $x = 0$, is

$$\bar{V} = \frac{Ee^{-qx}}{p}. \tag{4}$$

And, by § 83 (4), the transform of the current is

$$\bar{I} = E\sqrt{\left(\frac{Cp+G}{Lp+R}\right)}\frac{e^{-qx}}{p}. \tag{5}$$

Therefore, using the Inversion Theorem,

$$V = \frac{E}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t-\mu x}\,\frac{d\lambda}{\lambda}, \tag{6}$$

$$I = \frac{E}{2\pi i}\sqrt{\left(\frac{C}{L}\right)}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\sqrt{\left(\frac{\lambda+2\beta}{\lambda+2\alpha}\right)}e^{\lambda t-\mu x}\,\frac{d\lambda}{\lambda}, \tag{7}$$

where $\mu = (1/v)[(\lambda+2\alpha)(\lambda+2\beta)]^{\frac{1}{2}}$.

The integrals (6) and (7) can be evaluated† in terms of the Bessel function of imaginary argument,‡ $I_n(z)$, by using the

† For an entirely different treatment see Jeffreys, loc. cit., p. 104. A discussion using the Inversion Theorem is given by McLachlan, *Math. Gazette*, **22** (1938), 37.

‡ $I$ has been used here for current in place of the engineer's $i$ to avoid confusion with $\sqrt{(-1)}$. The Bessel functions always have a suffix, so there will be no ambiguity.

integral representation† of this function, namely,

$$\left(\frac{2}{z}\right)^n I_n(z) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{u+z^2/4u}\,\frac{du}{u^{n+1}}, \quad n > -1. \tag{8}$$

We discuss first (6) and, to reduce it to a form similar to (8), make the substitution:

$$(\lambda+2\alpha)^{\frac{1}{2}}+(\lambda+2\beta)^{\frac{1}{2}} = \xi^{\frac{1}{2}},$$

so that
$$(\lambda+2\alpha)^{\frac{1}{2}}-(\lambda+2\beta)^{\frac{1}{2}} = 2\sigma\xi^{-\frac{1}{2}},$$

$$\left.\begin{aligned}\lambda &= \frac{1}{4}\left\{\xi+\frac{4\sigma^2}{\xi}-4\rho\right\},\\ v\mu &= [(\lambda+2\alpha)(\lambda+2\beta)]^{\frac{1}{2}} = \frac{1}{4}\left\{\xi-\frac{4\sigma^2}{\xi}\right\}\end{aligned}\right\} \tag{9}$$

and
$$\frac{d\xi}{\xi} = \frac{d\lambda}{[(\lambda+2\alpha)(\lambda+2\beta)]^{\frac{1}{2}}} = \frac{d\lambda}{v\mu}.$$

The path from $\gamma-i\infty$ to $\gamma+i\infty$ in the $\lambda$-plane transforms into one of the same type‡ in the $\xi$-plane, which we shall denote by $\int\limits_{\gamma'-i\infty}^{\gamma'+i\infty}$. If we apply the transformation (9) directly to (6) no

† *W.B.F.*, §§ 6.2, 6.22, or *G. and M.*, p. 53 (43). *Formally* the result may be derived as follows: since $t^m/m!$ is the Laplace Transform of $p^{-m-1}$ (when $m > -1$), we have, by the Inversion Theorem,

$$\frac{t^m}{m!} = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\,\frac{d\lambda}{\lambda^{m+1}}, \quad m > -1.$$

Now
$$I_n(z) = \sum_{r=0}^{\infty}\frac{(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!}.$$

Introducing in this the result above, with $t = 1$, and $n+r$ written for $m$, and assuming we may invert the orders of integration and summation, we have

$$I_n(z) = \sum_{r=0}\frac{(\frac{1}{2}z)^{n+2r}}{2\pi i r!}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda}\,\frac{d\lambda}{\lambda^{n+r+1}}$$

$$= \frac{1}{2\pi i}(\tfrac{1}{2}z)^n\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda}\,\frac{d\lambda}{\lambda^{n+1}}\sum_{r=0}^{\infty}\frac{(\frac{1}{2}z)^{2r}}{\lambda^r r!} = \frac{(\frac{1}{2}z)^n}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda+z^2/4\lambda}\,\frac{d\lambda}{\lambda^{n+1}}.$$

For a complete proof, see references above.

‡ It is, of course, not a straight line but can be deformed into the line $(\gamma'-i\infty,\ \gamma'+i\infty)$ by Cauchy's theorem.

simplification will result, but the last result of (9) suggests that we evaluate first

$$X = \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t-\mu x}}{v\mu}\, d\lambda, \tag{10}$$

which becomes, on making the substitution (9),

$$X = \frac{1}{2\pi i} \int\limits_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{d\xi}{\xi} \exp\left\{-\rho t + \tfrac{1}{4}\xi\left(t-\frac{x}{v}\right)+\frac{\sigma^2}{\xi}\left(t+\frac{x}{v}\right)\right\}. \tag{11}$$

If $t > x/v$, putting $\frac{1}{4}\xi(t-x/v) = u$ in (11) we have from (8) with $n = 0$:

$$X = e^{-\rho t} I_0\left[\sigma \sqrt{\left(t^2-\frac{x^2}{v^2}\right)}\right], \quad t > \frac{x}{v}. \tag{12}$$

For the case $t < x/v$ consider the integral of the integrand of (11) taken round the contour of Fig. 17, consisting of portion of the contour $(\gamma'-i\infty, \gamma'+i\infty)$ completed to the right by portion of a circle of radius $R$ and centre the origin. The integrand is regular inside and on this contour and has no poles within it. Thus, by Cauchy's theorem, the integral round the contour is zero. It is easy to show that in the limit $R \to \infty$ the integral round the circular arc vanishes. Thus we are left with

$$X = 0, \qquad t < \frac{x}{v}. \tag{13}$$

Combining (12) and (13),

$$X = \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t-\mu x}}{v\mu}\, d\lambda = e^{-\rho t} I_0\left[\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right] H\left(t-\frac{x}{v}\right), \tag{14}$$

where $H(t)$ is defined in § 88 (2).

Integrating (14) with respect to $t$ from 0 to $t$ gives, assuming†

† These operations and also those leading to (16) may most easily be justified by using the paths $L$ and $L'$ of § 58. It is easy to show that the integrals round the portions $BB''$ and $AA''$ of the large circle in Fig. 15 vanish in the limit as $R = \infty$, and thus the path $L$ $(\gamma-i\infty, \gamma+i\infty)$ can be transformed into the path $L'$. Then all integrals concerned along $L'$ can be shown to be uniformly convergent. Also, in this way, it may be verified that (6) and (7) satisfy the differential equation (for $t \gtrless x/v$). The proofs follow the lines of the example given in § 58.

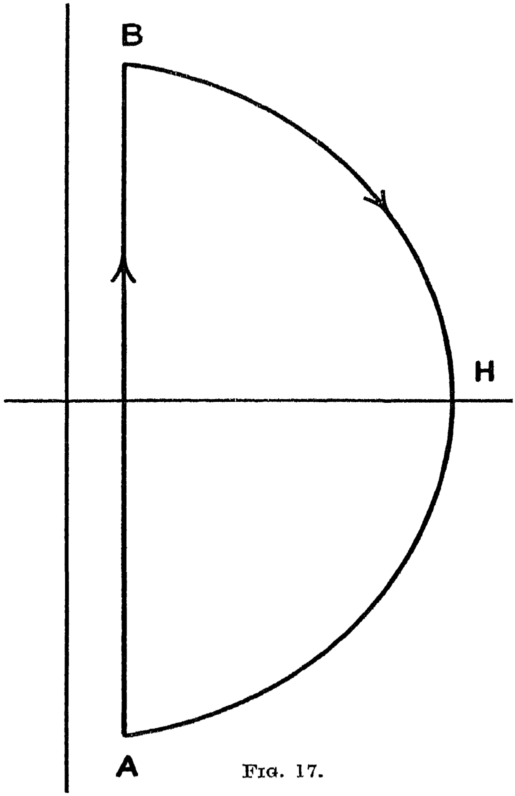


FIG. 17.

that we can invert the orders of integration on the left-hand side,†

$$\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t-\mu x}}{v\lambda\mu}\,d\lambda = \int\limits_{0}^{t} e^{-\rho\tau}I_0\left[\sigma\left(\tau^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]H\left(\tau-\frac{x}{v}\right)d\tau$$

$$\left.\begin{aligned} &= \int\limits_{x/v}^{t} e^{-\rho\tau}I_0\left[\sigma\left(\tau^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]d\tau, \quad t > x/v, \\ &= 0, \quad t < x/v. \end{aligned}\right\} \tag{15}$$

† Using Fig. 17 it can be shown that

$$\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{-\mu x}}{v\lambda\mu}\,d\lambda = 0.$$

Differentiating (15) with respect to $x$, assuming that the orders of differentiation and integration on the left-hand side may be interchanged, gives

$$\left.\begin{aligned}&\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{d\lambda}{\lambda}e^{\lambda t-\mu x}\\ &\quad = e^{-\rho x/v}+\frac{\sigma x}{v}\int\limits_{x/v}^{t}e^{-\rho\tau}\frac{I_1[\sigma\sqrt{(\tau^2-x^2/v^2)}]}{\sqrt{(\tau^2-x^2/v^2)}}\,d\tau, \quad t>x/v,\\ &\quad = 0, \quad t<x/v,\end{aligned}\right\} \tag{16}$$

where we have used the results

$$\frac{d}{dx}I_0(x)=I_1(x), \tag{17}$$

and

$$I_0(0)=1. \tag{18}$$

The left-hand side of (16) is the integral required for $V$ in (6). Therefore finally

$$V=E\left\{e^{-\rho x/v}+\frac{\sigma x}{v}\int\limits_{x/v}^{t}e^{-\rho\tau}\frac{I_1[\sigma(\tau^2-x^2/v^2)^{\frac{1}{2}}]}{(\tau^2-x^2/v^2)^{\frac{1}{2}}}\,d\tau\right\}H\left(t-\frac{x}{v}\right). \tag{19}$$

Thus the potential at the point $x$ is zero until time $t=x/v$, when the disturbance reaches it; it then jumps to $Ee^{-\rho x/v}$ and varies according to (19), its final value as $t\to\infty$ being†

$$Ee^{-\rho x/v}+\frac{E\sigma x}{v}\int\limits_{v/x}^{\infty}e^{-\rho\tau}\frac{I_1[\sigma(\tau^2-x^2/v^2)^{\frac{1}{2}}]}{(\tau^2-x^2/v^2)^{\frac{1}{2}}}\,d\tau = Ee^{-(2x/v)\sqrt{(\alpha\beta)}}=Ee^{-x\sqrt{(RG)}}.$$

† From the Inversion Theorem and (14) we have

$$\frac{e^{-\mu x}}{\mu v}=\int\limits_0^\infty e^{-\rho t-\lambda t}I_0\left[\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]H\left(t-\frac{x}{v}\right)\,dt=\int\limits_{x/v}^{\infty}e^{-t(\rho+\lambda)}I_0\left[\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]\,dt. \tag{20}$$

Differentiating with respect to $x$ gives, using (17) and (18),

$$e^{-\mu x}=e^{-(x/v)(\rho+\lambda)}+\frac{\sigma x}{v}\int\limits_{x/v}^{\infty}e^{-t(\rho+\lambda)}\frac{I_1[\sigma(t^2-x^2/v^2)^{\frac{1}{2}}]}{(t^2-x^2/v^2)^{\frac{1}{2}}}\,dt,$$

since the last integral is uniformly convergent. Also both sides are continuous functions of $\lambda$; so letting $\lambda\to 0$,

$$e^{-(2x/v)\sqrt{(\alpha\beta)}}-e^{-\rho x/v}=\frac{\sigma x}{v}\int\limits_{x/v}^{\infty}e^{-\rho t}\frac{I_1[\sigma(t^2-x^2/v^2)^{\frac{1}{2}}]}{(t^2-x^2/v^2)^{\frac{1}{2}}}\,dt.$$

*To determine the current* we have, by (7),

$$I = \frac{E}{2\pi i}\sqrt{\left(\frac{C}{L}\right)} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{\lambda+2\beta}{v\lambda\mu} e^{\lambda t-\mu x}\, d\lambda.$$

Therefore

$$I = E\sqrt{\left(\frac{C}{L}\right)}\left\{e^{-\rho t}I_0\left[\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]+\right.$$

$$\left.+2\beta\int\limits_{x/v} e^{-\rho\tau}I_0\left[\sigma\left(\tau^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]\, d\tau\right\}H\left(t-\frac{x}{v}\right), \quad (21)$$

using (14) and (15).

Thus $I = 0$ for $t < x/v$; at $t = x/v$ it jumps to $E\sqrt{(C/L)}e^{-\rho x/v}$, and† as $t \to \infty$, $I \to E\sqrt{(\beta C/\alpha L)}e^{-(2x/v)\sqrt{(\alpha\beta)}} = E\sqrt{(G/R)}e^{-x\sqrt{(RG)}}$.

**91.** *Uniform semi-infinite transmission line, $x > 0$, with zero initial current and charge. E.M.F. $f(t)$ applied at $x = 0$, for $t > 0$.*

Let $\bar{f}(p) = \int\limits_0^\infty e^{-pt}f(t)\, dt$, then, proceeding as in § 90, we obtain in place of (6) and (7),

$$V = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t-\mu x}\bar{f}(\lambda)\, d\lambda, \quad (1)$$

$$I = \frac{1}{2\pi i}\sqrt{\left(\frac{C}{L}\right)}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\sqrt{\left(\frac{\lambda+2\beta}{\lambda+2\alpha}\right)}e^{\lambda t-\mu x}\bar{f}(\lambda)\, d\lambda. \quad (2)$$

To evaluate (1) we notice that using § 90 (14) and § 3, Theorem VI,

† Putting $\lambda = 0$ in (20) gives

$$\int\limits_{x/v}^{\infty} e^{-\rho t}I_0\left[\sigma\left(t^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]\, dt = \frac{1}{2\sqrt{(\alpha\beta)}}e^{-(2x/v)\sqrt{(\alpha\beta)}}.$$

Also $\lim\limits_{t\to\infty} e^{-\rho t}I_0[\sigma(t^2-x^2/v^2)^{\frac{1}{2}}] = 0$, since $\rho > \sigma$.

$$\left.\begin{aligned}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t-\mu x}}{v\mu}\bar{f}(\lambda)\,d\lambda &= \int\limits_0^t f(t-\tau)e^{-\rho\tau}I_0\left[\sigma\left(\tau^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]H\left(\tau-\frac{x}{v}\right)d\tau\\ &= \int\limits_{x/v}^t f(t-\tau)e^{-\rho\tau}I_0\left[\sigma\left(\tau^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]d\tau, \quad t>x/v,\\ &= 0, \quad t<x/v.\end{aligned}\right\} \tag{3}$$

Differentiating† (3) with respect to $x$ brings the left-hand side to the form (1); thus, using § 90 (17) and (18), we have

$$\begin{aligned}V &= \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t-\mu x}\bar{f}(\lambda)\,d\lambda\\ &= f\left(t-\frac{x}{v}\right)e^{-\rho x/v}+\frac{\sigma x}{v}\int\limits_{x/v}^t f(t-\tau)e^{-\rho\tau}\frac{I_1[\sigma(\tau^2-x^2/v^2)^{\frac{1}{2}}]}{(\tau^2-x^2/v^2)^{\frac{1}{2}}}\,d\tau, \quad t>x/v,\\ &= 0, \quad t<x/v.\end{aligned} \tag{4}$$

*To find the current*: Differentiate (4) with respect to $\alpha$, using $\rho=\alpha+\beta$, $\sigma=\alpha-\beta$, $\mu=(1/v)[(\lambda+2\alpha)(\lambda+2\beta)]^{\frac{1}{2}}$, and we obtain

$$\begin{aligned}-\frac{1}{2\pi i}\frac{x}{v}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}&\frac{(\lambda+2\beta)e^{\lambda t-\mu x}\bar{f}(\lambda)\,d\lambda}{\mu v}\\ &= -\frac{x}{v}f\left(t-\frac{x}{v}\right)e^{-\rho x/v}+\frac{x}{v}\int\limits_{x/v}^t\left\{f(t-\tau)e^{-\rho\tau}\frac{I_1[\sigma(\tau^2-x^2/v^2)^{\frac{1}{2}}]}{(\tau^2-x^2/v^2)^{\frac{1}{2}}}-\right.\\ &\qquad -\sigma\tau f(t-\tau)e^{-\rho\tau}\frac{I_1[\sigma(\tau^2-x^2/v^2)^{\frac{1}{2}}]}{(\tau^2-x^2/v^2)^{\frac{1}{2}}}+\\ &\qquad \left.+\sigma f(t-\tau)e^{-\rho\tau}I_1'\left[\sigma\left(\tau^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]\right\}d\tau, \quad t>x/v,\\ &= 0, \quad t<x/v.\end{aligned}$$

† Assuming that the integral can be differentiated under the integral sign, which implies conditions on $f(t)$.

Thus, using (2) and the result $zI_1'(z)+I_1(z) = zI_0(z)$, we have, for $t > x/v$,

$$\sqrt{\left(\frac{C}{L}\right)}\left\{f\left(t-\frac{x}{v}\right)e^{-px/v}+\right.$$

$$\left.+\int\limits_{x/v}^{t} f(t-\tau)e^{-p\tau}\left\{\frac{\sigma\tau I_1[\sigma(\tau^2-x^2/v^2)^{\frac{1}{2}}]}{(\tau^2-x^2/v^2)^{\frac{1}{2}}}-\sigma I_0\left[\sigma\left(\tau^2-\frac{x^2}{v^2}\right)^{\frac{1}{2}}\right]\right\}d\tau.\right. \qquad (5)$$

If $\sigma = 0$, the integrals in (4) and (5) disappear, and the potential at $x$ follows accurately that at $x = 0$ with a time lag $x/v$ and a constant diminution in amplitude. Hence the term 'distortionless' line.

**92.** *A doubly-infinite line, $-\infty < x < \infty$, with initial potential $V^{(0)} = f(x)$, and initial current $I^{(0)} = g(x)$.*

The subsidiary equation § 83 (3) is

$$\frac{\quad}{dx^2}-q^2\overline{V} = Lg'(x)-C(Lp+R)f(x), \qquad (1)$$

where, in the notation of § 90 (3), $q^2 = (1/v^2)(p+2\alpha)(p+2\beta)$.

We solve (1) by finding the Green's function† $G(x,\xi)$ for the differential equation

$$\frac{d^2\overline{V}}{dx^2}-q^2\overline{V} = 0, \qquad (2)$$

and boundary conditions,

$$\overline{V} \text{ finite} \quad \text{as } x \to \pm\infty. \qquad (3)$$

Let it be

$$\left.\begin{aligned} G(x,\xi) &= Ae^{-qx}, \quad x > \xi, \\ &= Be^{qx}, \quad x < \xi. \end{aligned}\right\} \qquad (4)$$

This will satisfy (2) and (3). It has also to satisfy the conditions

$$G(x,\xi) \quad \text{is to be continuous at} \quad x = \xi,$$

and

$$\left[\frac{\partial G(x,\xi)}{\partial x}\right]_{\xi-0}^{\xi+0} = 1.$$

† See § 68. Variation of parameters may also be used.

Substituting from (4) these give

$$\left.\begin{aligned} Ae^{-q\xi}-Be^{q\xi} &= 0, \\ -Ae^{-q\xi}-Be^{q\xi} &= 1/q. \end{aligned}\right\}$$

Thus $$A = -\frac{1}{2q}e^{q\xi}, \qquad B = -\frac{1}{2q}e^{-q\xi},$$

and $$\left.\begin{aligned} G(x,\xi) &= -\frac{1}{2q}e^{-q(x-\xi)}, \quad x > \xi, \\ &= -\frac{1}{2q}e^{q(x-\xi)}, \quad x < \xi. \end{aligned}\right\} \tag{5}$$

Thus, as in § 68, the solution of the non-homogeneous equation (1) and boundary conditions (3) is

$$\begin{aligned} \bar{V}(x) &= \int\limits_{-\infty}^{\infty} G(\xi,x)[Lg'(\xi)-C(Lp+R)f(\xi)]\,d\xi \\ &= \frac{1}{2q}\int\limits_{-\infty}^{x} [C(Lp+R)f(\xi)-Lg'(\xi)]e^{q(\xi-x)}\,d\xi + \\ &\qquad + \frac{1}{2q}\int\limits_{x}^{\infty} [C(Lp+R)f(\xi)-Lg'(\xi)]e^{-q(\xi-x)}\,d\xi. \end{aligned}$$

Putting $\xi = x-v\eta$ in the first, and $\xi = x+v\eta$ in the second of these, we get

$$\begin{aligned} \bar{V}(x) &= \frac{v}{2q}\int\limits_{0}^{\infty} \{C(Lp+R)[f(x-v\eta)+f(x+v\eta)]- \\ &\qquad -L[g'(x-v\eta)+g'(x+v\eta)]\}e^{-qv\eta}\,d\eta \\ &= \frac{v}{2q}\int\limits_{0}^{\infty} [RCf(x-v\eta)+RCf(x+v\eta)- \\ &\qquad -Lg'(x-v\eta)-Lg'(x+v\eta)]e^{-qv\eta}\,d\eta + \\ &\qquad + \frac{LCpv}{2q}\int\limits^{\infty} [f(x-v\eta)+f(x+v\eta)]e^{-qv\eta}\,d\eta \\ &= \bar{V}_1(x)+\bar{V}_2(x), \quad \text{say.} \end{aligned} \tag{6}$$

We consider the two parts separately. Taking first $\bar{V}_1(x)$,

applying the Inversion Theorem, and assuming† that the orders of the $\lambda$ and $\eta$ integrations may be interchanged, we have

$$V_1(x,t) = \frac{v}{4\pi i}\int\limits_0^\infty [RCf(x-v\eta)+RCf(x+v\eta)-$$

$$-Lg'(x-v\eta)-Lg'(x+v\eta)]\,d\eta \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t-\mu v\eta}}{\mu}\,d\lambda, \qquad (7)$$

where $\mu = (1/v)[(\lambda+2\alpha)(\lambda+2\beta)]^{\frac{1}{2}}$.

The contour integral in (7) has been evaluated in § 90 (14), so, inserting this value in (7), we find

$$V_1(x,t) = \tfrac{1}{2}v^2\int\limits_0^\infty [RCf(x-v\eta)+RCf(x+v\eta)-$$

$$-Lg'(x-v\eta)-Lg'(x+v\eta)]e^{-\rho t}I_0[\sigma(t^2-\eta^2)^{\frac{1}{2}}]H(t-\eta)\,d\eta$$

$$= \tfrac{1}{2}v^2e^{-\rho t}\int\limits_0^t [RCf(x-v\eta)-Lg'(x-v\eta)]I_0[\sigma(t^2-\eta^2)^{\frac{1}{2}}]\,d\eta +$$

$$+\tfrac{1}{2}v^2e^{-\rho t}\int\limits^t [RCf(x+v\eta)-Lg'(x+v\eta)]I_0[\sigma(t^2-\eta^2)^{\frac{1}{2}}]\,d\eta.$$

Putting $x-v\eta = \xi$ in the first of these, and $x+v\eta = \xi$ in the second, we have finally

$$V_1(x,t) = \tfrac{1}{2}ve^{-\rho t}\int\limits_{x-vt}^{x+vt} [RCf(\xi)-Lg'(\xi)]I_0\left[\sigma\left(t^2-\frac{(x-\xi)^2}{v^2}\right)^{\frac{1}{2}}\right]\,d\xi. \qquad (8)$$

To evaluate $V_2(x,t)$, we have from (6) and the Inversion Theorem

$$V_2(x,t) = \frac{LCv}{4\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{\lambda e^{\lambda t}\,d\lambda}{\mu}\int\limits_0^\infty [f(x-v\eta)+f(x+v\eta)]e^{-v\eta\mu}\,d\eta.$$

† To justify this, conditions must be imposed on $f(x)$ and $g(x)$. The same remark applies to the operations involved in evaluating $V_2(x,t)$.

Integrating this equation with respect to $t$ from 0 to $t$ gives

$$\int_0^t V_2(x,\tau)\,d\tau = \frac{LCv}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}\,d\lambda}{\mu}\int_0^{\infty}[f(x-v\eta)+f(x+v\eta)]e^{-\eta\mu v}\,d\eta$$

$$= \frac{LCv}{4\pi i}\int_0^{\infty}[f(x-v\eta)+f(x+v\eta)]\,d\eta\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t-\eta\mu v}}{\mu}\,d\lambda. \tag{9}$$

The contour integral in (9) has been evaluated in § 90 (14), so, inserting this value,

$$\int_0^t V_2(x,\tau)\,d\tau$$

$$= \tfrac{1}{2}LCv^2e^{-\rho t}\int_0^{\infty}[f(x-v\eta)+f(x+v\eta)]I_0[\sigma(t^2-\eta^2)^{\frac{1}{2}}]H(t-\eta)\,d\eta$$

$$= \tfrac{1}{2}e^{-\rho t}\int_0^{t}[f(x-v\eta)+f(x+v\eta)]I_0[\sigma(t^2-\eta^2)^{\frac{1}{2}}]\,d\eta$$

$$= \frac{1}{2v}e^{-\rho t}\int_{x-vt}^{x+vt}f(\xi)I_0\left\{\sigma\left[t^2-\frac{(x-\xi)^2}{v^2}\right]^{\frac{1}{2}}\right\}\,d\xi.$$

Differentiating this with respect to $t$, and using the results $I_0(0)=1$, $I_0'(x)=I_1(x)$, we have

$$V_2(x,t) = \tfrac{1}{2}e^{-\rho t}[f(x+vt)+f(x-vt)]-$$

$$-\frac{\rho}{2v}e^{-\rho t}\int_{x-vt}^{x+vt}f(\xi)I_0\left\{\sigma\left[t^2-\frac{(x-\xi)^2}{v^2}\right]^{\frac{1}{2}}\right\}\,d\xi+$$

$$+\frac{\sigma t}{2v}e^{-\rho t}\int_{x-vt}^{x+vt}f(\xi)\frac{I_1\{\sigma[t^2-(x-\xi)^2/v^2]^{\frac{1}{2}}\}}{[t^2-(x-\xi)^2/v^2]^{\frac{1}{2}}}\,d\xi. \tag{10}$$

Adding (8) and (10),

$$V(x,t) = \tfrac{1}{2}e^{-\rho t}[f(x+vt)+f(x-vt)]+$$

$$+\frac{1}{2v}e^{-\rho t}\int_{x-vt}^{x+vt}\left[\sigma f(\xi)-\frac{1}{C}g'(\xi)\right]I_0\left\{\sigma\left[t^2-\frac{(x-\xi)^2}{v^2}\right]^{\frac{1}{2}}\right\}\,d\xi+$$

$$+\frac{\sigma t}{2v}e^{-\rho t}\int_{x-vt}^{x+vt}f(\xi)\frac{I_1\{\sigma[t^2-(x-\xi)^2/v^2]^{\frac{1}{2}}\}}{[t^2-(x-\xi)^2/v^2]^{\frac{1}{2}}}\,d\xi. \tag{11}$$

In the 'distortionless' case, $\sigma = 0$, this becomes

$$V(x,t) = \tfrac{1}{2}e^{-\rho t}[f(x+vt)+f(x-vt)]-\frac{1}{2vC}e^{-\rho t}\int\limits_{x-vt}^{x+vt} g'(\xi)\,d\xi. \tag{12}$$

**93.** *The finite line with none of R, L, G, C zero.*

In § 88 a method was developed for expressing solutions in terms of successively reflected waves, and this was applied to the pure wave cases $R = G = 0$ and $R/L = G/C$. It may now be applied to the general case.

For the problem of § 88 we have, by § 88 (4),

$$\bar{V} = \frac{E}{p}[e^{-qx}+e^{-q(2l-x)}-e^{-q(2l+x)}-e^{-q(4l-x)}+...], \tag{1}$$

where $q = (1/v)[(p+2\alpha)(p+2\beta)]^{\frac{1}{2}}$.

The function whose transform is $(1/p)e^{-qx}$ has been found in § 90 (16); using this result in the terms of (1), we have

$$V = E\left\{e^{-\rho x/v}+\frac{\sigma x}{v}\int\limits_{x/v}^{t} e^{-\rho\tau}\frac{I_1[\sigma(\tau^2-x^2/v^2)^{\frac{1}{2}}]}{(\tau^2-x^2/v^2)^{\frac{1}{2}}}\,d\tau\right\}H\left(t-\frac{x}{v}\right)+$$

$$+E\left\{e^{-\rho(2l-x)/v}+\frac{\sigma(2l-x)}{v}\int\limits_{(2l-x)/v}^{t} e^{-\rho\tau}\frac{I_1[\sigma(\tau^2-(2l-x)^2/v^2)^{\frac{1}{2}}]}{[\tau^2-(2l-x)^2/v^2]^{\frac{1}{2}}}\,d\tau\right\}\times$$

$$\times H\left(t-\frac{2l-x}{v}\right)-....$$

Thus, in addition to the terms of the first column which represent the arrival of successively reflected waves, there are the terms of the second column representing 'tails' of these waves.

The case in which the E.M.F. applied at $x = 0$ is an arbitrary function of the time can be dealt with in the same way using § 91 (4) in the place of § 90 (16).

**94.** *Non-uniform lines.*

If the parameters $R$, $L$, $G$, $C$ of the line are functions of $x$, equations (1) and (2) of § 83 still hold, but in place of (3) we shall have for $\bar{V}$ a linear second-order differential equation whose coefficients are functions of $x$.

As an example consider Heaviside's 'Bessel cable' in which

$$L = L'/x, \qquad R = R'/x, \qquad C = C'x, \qquad G = G'x,$$

where $L'$, $R'$, $C'$, $G'$ are constants. Then, if $I^{(0)}(x) = V^{(0)}(x) = 0$, equations (2) of § 83 become

$$(L'p+R')\frac{\bar{I}}{x} = -\frac{d\bar{V}}{dx},$$

$$(C'p+G')x\bar{V} = -\frac{d\bar{I}}{dx}.$$

Hence
$$\frac{d^2\bar{V}}{dx^2}+\frac{1}{x}\frac{d\bar{V}}{dx}-q'^2\bar{V} = 0, \tag{1}$$

where $q'^2 = (L'p+R')(C'p+G')$. Also $\bar{I} = -\frac{x}{L'p+R'}\frac{d\bar{V}}{dx}$.

The general solution of (1) is $\bar{V} = AI_0(q'x)+BK_0(q'x)$, where $A$ and $B$ are to be found from the terminal conditions. If $L' = G' = 0$, the problem is of the same type as those on radial flow of heat in circular cylinders discussed in Chapter VI.

CHAPTER X

# ELECTRIC WAVE AND DIFFUSION PROBLEMS

**95.** The Maxwell equations for a uniform isotropic medium† of dielectric constant $\kappa$, permeability $\mu$, and conductivity $\sigma$, are

$$\operatorname{curl}\mathbf{H} = \frac{4\pi\sigma}{c}\mathbf{E}+\frac{\kappa}{c}\frac{\partial\mathbf{E}}{\partial t}, \tag{1}$$

$$\operatorname{curl}\mathbf{E} = -\frac{\mu}{c}\frac{\partial\mathbf{H}}{\partial t}, \tag{2}$$

$$\operatorname{div}\mathbf{E} = \frac{4\pi\rho}{\kappa}, \tag{3}$$

$$\operatorname{div}\mathbf{H} = 0, \tag{4}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field strengths, and $\rho$ is the volume density of free electricity. $\mathbf{E}$, $\kappa$, $\rho$, and $\sigma$ are measured in electrostatic units, and $\mathbf{H}$ and $\mu$ in electromagnetic units.

Multiply these equations by $e^{-pt}$, $p > 0$, and integrate with respect to $t$ from 0 to $\infty$. Then, if $\mathbf{E}^{(0)}$ and $\mathbf{H}^{(0)}$ are the values of $\mathbf{E}$ and $\mathbf{H}$ when $t = 0$, and we write $\overline{\mathbf{E}}$ and $\overline{\mathbf{H}}$ for the vectors whose components are the Laplace Transforms of the components of $\mathbf{E}$ and $\mathbf{H}$, we have from (1) to (4) subsidiary equations

$$\operatorname{curl}\overline{\mathbf{H}} = \frac{4\pi\sigma}{c}\overline{\mathbf{E}}+\frac{\kappa p}{c}\overline{\mathbf{E}}-\frac{\kappa}{c}\mathbf{E}^{(0)}, \tag{5}$$

$$\operatorname{curl}\overline{\mathbf{E}} = -\frac{\mu p}{c}\overline{\mathbf{H}}+\frac{\mu}{c}\mathbf{H}^{(0)}, \tag{6}$$

$$\operatorname{div}\overline{\mathbf{E}} = \frac{4\pi}{\kappa}\bar{\rho}, \tag{7}$$

$$\operatorname{div}\overline{\mathbf{H}} = 0. \tag{8}$$

† The medium is supposed not to contain discontinuities of electric or magnetic force. For a discussion of cases in which it does, see Bromwich, *Proc. Lond. Math. Soc.* (2), **28** (1927), 438.

From these we obtain, on eliminating $\overline{\mathbf{E}}$ and $\overline{\mathbf{H}}$ respectively,

$$\nabla^2\overline{\mathbf{H}}-q^2\overline{\mathbf{H}} = -\frac{q^2}{p}\mathbf{H}^{(0)}+\frac{\kappa}{c}\operatorname{curl}\mathbf{E}^{(0)}, \tag{9}$$

$$\nabla^2\overline{\mathbf{E}}-q^2\overline{\mathbf{E}} = \frac{\kappa\mu p}{q^2c^2}\operatorname{grad}\operatorname{div}\mathbf{E}^{(0)}-\frac{\kappa\mu}{c^2}p\mathbf{E}^{(0)}-\frac{\mu}{c}\operatorname{curl}\mathbf{H}^{(0)}, \tag{10}$$

where
$$q^2 = \frac{4\pi\sigma\mu}{c^2}p+\frac{\kappa\mu}{c^2}p^2 \tag{11}$$

The general solution of (9) and (10) may be obtained by the methods of Potential Theory, and $\mathbf{E}$ and $\mathbf{H}$ then follow on applying the Inversion Theorem. For an application of this method of procedure in a similar problem see § 105, also Bromwich, loc. cit. In particular cases, however, it is usually easier to write down equations (1)–(4) in the appropriate curvilinear coordinates and to solve directly the resulting subsidiary equations.

In the following sections we shall confine ourselves to two simpler cases:

(i) a non-conducting dielectric

$$\sigma = 0, \qquad q^2 = \frac{\kappa\mu}{c^2}p^2; \tag{12}$$

(ii) a good (metallic) conductor in which the displacement current may be neglected in comparison with the conduction current. Here

$$q^2 = \frac{4\pi\sigma\mu}{c^2}p. \tag{13}$$

**96.** *A flat conducting plate,* $-a < x < a$, $-\infty < y < \infty$, $-\infty < z < \infty$, *is initially free from electric and magnetic fields. At* $t = 0$ *a magnetic field* $H_0 \cos\omega t$, *parallel to the z-axis, is established at both faces* $x = \pm a$.

Here, since all quantities are independent of $y$ and $z$, the subsidiary equation is, by† § 95 (9),

$$\frac{d^2\overline{H}_z}{dx^2}-q^2\overline{H}_z = 0, \quad -a < x < a, \tag{1}$$

where
$$q^2 = p/k \quad \text{and} \quad k = \frac{}{4\pi\sigma\mu}. \tag{2}$$

† This may, of course, easily be obtained from first principles without using the general equation of § 95.

This is to be solved with boundary conditions

$$\bar{H}_z = \frac{pH_0}{p^2+\omega^2}, \quad \text{when } x = \pm a.$$

The solution is
$$\bar{H}_z = \frac{pH_0}{p^2+\omega^2}\frac{\cosh qx}{\cosh qa}, \tag{3}$$

therefore
$$H_z = \frac{H_0}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{\cosh \nu x}{\cosh \nu a}\frac{\lambda\, d\lambda}{\lambda^2+\omega^2}, \tag{4}$$

where† $\nu = \sqrt{(\lambda/k)}$.

The integrand is a single-valued function of $\lambda$. Proceeding in the usual way, using Fig. 10, we find that the integral in (4) may be replaced by $2i\pi$ times the sum of the residues at its poles. These poles are $\lambda = \pm i\omega$ and $\lambda = -k(n+\frac{1}{2})^2\pi^2/a^2$, $n = 0, 1,\ldots$.

The residue at $\lambda = -k(n+\frac{1}{2})^2\pi^2/a^2$ is

$$\frac{4k^2\pi^3(2n+1)^3(-1)^{n+1}}{k^2(2n+1)^4\pi^4+16\omega^2a^4}e^{-k(n+\frac{1}{2})^2\pi^2t/a^2}\cos(n+\tfrac{1}{2})\frac{\pi x}{a}. \tag{5}$$

The residue at $\lambda = i\omega$ is

$$\tfrac{1}{2}e^{i\omega t}\frac{\cosh x(i\omega/k)^{\frac{1}{2}}}{\cosh a(i\omega/k)^{\frac{1}{2}}} = \tfrac{1}{2}e^{i\omega t}\frac{\cosh \omega' x(1+i)}{\cosh \omega' a(1+i)},$$

where $\omega' = (\omega/2k)^{\frac{1}{2}}$.

This reduces to
$$\tfrac{1}{2}\beta e^{i(\omega t-\gamma)},$$

where
$$\beta = \left(\frac{\cosh 2\omega' x+\cos 2\omega' x}{\cosh 2\omega' a+\cos 2\omega' a}\right)^{\frac{1}{2}}, \tag{6}$$

and

$$\tan\gamma = \frac{\sinh \omega'(a-x)\sin \omega'(a+x)+\sinh \omega'(a+x)\sin \omega'(a-x)}{\cosh \omega'(a-x)\cos \omega'(a+x)+\cosh \omega'(a+x)\cos \omega'(a-x)}. \tag{7}$$

Thus, finally,

$$H_z = 4k^2\pi^3H_0\sum_{n=0}^{\infty}\frac{(2n+1)^3(-1)^{n+1}}{k^2(2n+1)^4\pi^4+16\omega^2a^4}\cos(n+\tfrac{1}{2})\frac{\pi x}{a}e^{-k(n+\frac{1}{2})^2\pi^2t/a^2}+ +\beta H_0\cos(\omega t-\gamma),$$

† Throughout this chapter $\nu$ will be used in place of the usual $\mu$ for $\sqrt{(\lambda/k)}$ to avoid confusion with the permeability.

where $\beta$ and $\gamma$ are given by (6) and (7). The last term is the usual expression for the steady state magnetic field.†

**97.** *The expressions for the Maxwell equations in cylindrical and spherical polar coordinates*‡ *are collected here for reference.*

I. *Cylindrical coordinates.* (1) and (2) of § 95 become

$$\frac{\partial H_z}{r\,\partial\theta}-\frac{\partial H_\theta}{\partial z}=\left(\frac{4\pi\sigma}{c}+\frac{\kappa}{c}\frac{\partial}{\partial t}\right)E_r, \tag{1}$$

$$\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}=\left(\frac{4\pi\sigma}{c}+\frac{\kappa}{c}\frac{\partial}{\partial t}\right)E_\theta, \tag{2}$$

$$\frac{\partial(rH_\theta)}{r\,\partial r}-\frac{\partial H_r}{r\,\partial\theta}=\left(\frac{4\pi\sigma}{c}+\frac{\kappa}{c}\frac{\partial}{\partial t}\right)E_z, \tag{3}$$

$$\frac{\partial E_z}{r\,\partial\theta}-\frac{\partial E_\theta}{\partial z}=-\frac{\mu}{c}\frac{\partial H_r}{\partial t}, \tag{4}$$

$$\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}=-\frac{\mu}{c}\frac{\partial H_\theta}{\partial t}, \tag{5}$$

$$\frac{\partial(rE_\theta)}{r\,\partial r}-\frac{\partial E_r}{r\,\partial\theta}=-\frac{\mu}{c}\frac{\partial H_z}{\partial t}. \tag{6}$$

We shall require only the case in which all quantities are independent of $z$ so that the terms in the equations above involving derivatives with respect to $z$ disappear. Equations (1)–(6) then fall into two groups of three, namely, (1), (2), (6) containing $H_z$, $E_r$, and $E_\theta$, and (3), (4), (5) containing $H_r$, $H_\theta$, and $E_z$.

The subsidiary equations for the first set are

$$\frac{\partial \bar{H}_z}{r\,\partial\theta}=\left(\frac{4\pi\sigma}{c}+\frac{\kappa}{c}p\right)\bar{E}_r-\frac{\kappa}{c}E_r^{(0)}, \tag{7}$$

$$-\frac{\partial \bar{H}_z}{\partial r}=\left(\frac{4\pi\sigma}{c}+\frac{\kappa}{c}p\right)\bar{E}_\theta-\frac{\kappa}{c}E_\theta^{(0)}, \tag{8}$$

$$\frac{\partial(r\bar{E}_\theta)}{r\,\partial r}-\frac{1}{r}\frac{\partial \bar{E}_r}{\partial\theta}=-\frac{\mu}{c}p\bar{H}_z+\frac{\mu}{c}H_z^{(0)}. \tag{9}$$

† Russell, *Alternating Currents*, **1** (2nd ed., 1914), 495.

‡ For the change of variables see Weatherburn, *Advanced Vector Analysis* (1924), § 11.

Substituting from (7) and (8) in (9) gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \bar{H}_z}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2 \bar{H}_z}{\partial \theta^2}-q^2\bar{H}_z=\frac{\kappa}{c}\frac{\partial(rE_\theta^{(0)})}{r\,\partial r}-\frac{\kappa}{c}\frac{\partial E_r^{(0)}}{r\,\partial\theta}-\frac{q^2}{p}H_z^{(0)}, \tag{10}$$

where, as before, $$q^2=\frac{4\pi\sigma\mu}{c^2}p+\frac{\kappa\mu}{c^2}p^2. \tag{11}$$

When $\bar{H}_z$ has been found from (10), $\bar{E}_r$ and $\bar{E}_\theta$ are obtained by (7) and (8).

For the second set the subsidiary equations are

$$\frac{\partial(r\bar{H}_\theta)}{r\,\partial r}-\frac{\partial \bar{H}_r}{r\,\partial\theta}=\left(\frac{4\pi\sigma}{c}+\frac{\kappa}{c}p\right)\bar{E}_z-\frac{\kappa}{c}E_z^{(0)}, \tag{12}$$

$$\frac{\partial \bar{E}_z}{r\,\partial\theta}=-\frac{\mu}{c}p\bar{H}_r+\frac{\mu}{c}H_r^{(0)}, \tag{13}$$

$$-\frac{\partial \bar{E}_z}{\partial r}=-\frac{\mu}{c}p\bar{H}_\theta+\frac{\mu}{c}H_\theta^{(0)}. \tag{14}$$

Hence

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \bar{E}_z}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2 \bar{E}_z}{\partial \theta^2}-q^2\bar{E}_z=\frac{\mu}{c}\frac{\partial H_r^{(0)}}{r\,\partial\theta}-\frac{\mu}{c}\frac{\partial(rH_\theta^{(0)})}{r\,\partial r}-\frac{\kappa\mu}{c^2}pE_z^{(0)}. \tag{15}$$

II. In spherical polar coordinates (1) and (2) of § 95 become

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta H_\phi)-\frac{\partial H_\theta}{\partial\phi}\right]=\left(\frac{4\pi\sigma}{c}+\frac{\kappa}{c}\frac{\partial}{\partial t}\right)E_r, \tag{16}$$

$$\frac{1}{r\sin\theta}\left[\frac{\partial H_r}{\partial\phi}-\sin\theta\frac{\partial(rH_\phi)}{\partial r}\right]=\left(\frac{4\pi\sigma}{c}+\frac{\kappa}{c}\frac{\partial}{\partial t}\right)E_\theta, \tag{17}$$

$$\frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r}-\frac{\partial H_r}{\partial\theta}\right]=\left(\frac{4\pi\sigma}{c}+\frac{\kappa}{c}\frac{\partial}{\partial t}\right)E_\phi, \tag{18}$$

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\phi)-\frac{\partial E_\theta}{\partial\phi}\right]=-\frac{\mu}{c}\frac{\partial H_r}{\partial t}, \tag{19}$$

$$\frac{1}{r\sin\theta}\left[\frac{\partial E_r}{\partial\phi}-\sin\theta\frac{\partial(rE_\phi)}{\partial r}\right]=-\frac{\mu}{c}\frac{\partial H_\theta}{\partial t}, \tag{20}$$

$$\frac{1}{r}\left[\frac{\partial(rE_\theta)}{\partial r}-\frac{\partial E_r}{\partial\theta}\right]=-\frac{\mu}{c}\frac{\partial H_\phi}{\partial t}. \tag{21}$$

If the system is independent of $\phi$, these split up into two sets, namely, (16), (17), (21), containing $H_\phi$, $E_r$, $E_\theta$, and (18), (19), (20), containing $H_r$, $H_\theta$, $E_\phi$.

The subsidiary equations for the first of these sets are

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\bar{H}_\phi) = \left(\frac{4\pi\sigma}{c}+\frac{\kappa p}{c}\right)\bar{E}_r - \frac{\kappa}{c}E_r^{(0)}, \tag{22}$$

$$-\frac{1}{r}\frac{\partial(r\bar{H}_\phi)}{\partial r} = \left(\frac{4\pi\sigma}{c}+\frac{\kappa p}{c}\right)\bar{E}_\theta - \frac{\kappa}{c}E_\theta^{(0)}, \tag{23}$$

$$\frac{1}{r}\frac{\partial(r\bar{E}_\theta)}{\partial r} - \frac{1}{r}\frac{\partial\bar{E}_r}{\partial\theta} = -\frac{\mu p}{c}\bar{H}_\phi + \frac{\mu}{c}H_\phi^{(0)}, \tag{24}$$

whence, eliminating $\bar{E}_r$ and $\bar{E}_\theta$,

$$\frac{1}{r}\frac{\partial^2(r\bar{H}_\phi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\bar{H}_\phi)\right] - q^2\bar{H}_\phi$$
$$= -\frac{q^2}{p}H_\phi^{(0)} + \frac{\kappa}{rc}\frac{\partial(rE_\theta^{(0)})}{\partial r} - \frac{\kappa}{rc}\frac{\partial E_r^{(0)}}{\partial\theta}. \tag{25}$$

**98.** *A circular conducting cylinder $0 \leqslant r < a$ is initially free from electric and magnetic fields. At $t = 0$ a magnetic field $H_0 \cos\omega t$ is established outside the cylinder parallel to its axis.*†

The system is independent of both $\theta$ and $z$; so by § 97 (10) the subsidiary equation is

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\bar{H}_z}{dr}\right] - q^2\bar{H}_z = 0, \quad 0 \leqslant r < a, \tag{1}$$

where $$q = \sqrt{(p/k)} \quad \text{and} \quad k = \frac{c^2}{4\pi\mu\sigma}. \tag{2}$$

The boundary condition is

$$\bar{H} = \frac{pH_0}{p^2+\omega^2}, \quad \text{when } r = a. \tag{3}$$

This system of equations is exactly that discussed in § 54 with $H_z$ in place of $v$, and $k$ in place of $\kappa$. Thus, writing $\omega' = \sqrt{(\omega/k)}$, we have, using § 54 (7),

$$H_z = H_0\cos\omega t\frac{\operatorname{ber}\omega' r\operatorname{ber}\omega' a + \operatorname{bei}\omega' r\operatorname{bei}\omega' a}{\operatorname{ber}^2\omega' a + \operatorname{bei}^2\omega' a} +$$
$$+H_0\sin\omega t\frac{\operatorname{ber}\omega' r\operatorname{bei}\omega' a - \operatorname{bei}\omega' r\operatorname{ber}\omega' a}{\operatorname{ber}^2\omega' a + \operatorname{bei}^2\omega' a} +$$
$$+\frac{2k^2}{a}\sum_{s=1}^{\infty} e^{-k\alpha_s^2 t}\frac{J_0(\alpha_s r)}{J_0'(\alpha_s a)}\frac{\alpha_s^3}{k^2\alpha_s^4+\omega^2}, \tag{4}$$

† Cf. Russell, loc. cit., pp. 503 et seq.

where

$$\pm\alpha_s, \quad s = 1, 2,..., \quad \text{are the roots of} \quad J_0(\alpha a) = 0. \tag{5}$$

*Similarly, if the external magnetic field is constant,*† $H_0$, the result follows from § 53 (10):

$$H_z = H_0 + \frac{2H_0}{a}\sum_{s=1}^{\infty} e^{-k\alpha_s^2 t}\frac{J_0(a\alpha_s)}{\alpha_s J_0'(a\alpha_s)}, \tag{6}$$

where the $\alpha_s$ are the roots of (5).

**99.** *An infinite hollow conducting cylinder $a < r < b$ is free from electric and magnetic fields. At $t = 0$ a uniform magnetic field $H_0$ is established outside the cylinder and parallel to its axis.*

Since all quantities are independent of both $\theta$ and $z$, the subsidiary equation is, by § 97 (10),

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\bar{H}_z}{dr}\right] - q^2\bar{H}_z = 0, \quad a < r < b, \tag{1}$$

where $\qquad q^2 = p/k \quad \text{and} \quad k = c^2/(4\pi\sigma\mu). \qquad (2)$

Also, by § 97 (8), $\qquad \bar{E}_\theta = -\frac{c}{4\pi\sigma}\frac{d\bar{H}_z}{dr}. \qquad (3)$

The boundary condition at the outer surface is

$$\bar{H}_z = H_0/p, \quad \text{when } r = b. \tag{4}$$

That at the inner surface may be found‡ most easily as follows: Neglecting displacement currents, the magnetic field in the hollow interior is independent of $r$, and thus has the value at the inner surface $r = a$. Also $H_z$ and $E_\theta$ are continuous at $r = a$. Now § 95 (6) gives

$$\operatorname{curl}\bar{\mathbf{E}} = -\frac{p}{c}\bar{\mathbf{H}}, \quad r < a,$$

† The problem of the decay of magnetic field in a cylinder when the external field is removed may be treated in the same way. It is discussed in detail by Bromwich, *Proc. Lond. Math. Soc.* (2), **31** (1929), 209.

‡ It may also be obtained by a limiting process, e.g. regarding the interior $0 \leqslant r < a$ as having conductivity $\sigma_1$ and taking the limiting form of the boundary condition at $r = a$ as $\sigma_1 \to 0$.

and thus, taking the line integral of $\overline{\mathbf{E}}$ round the circle $r = a$, we find

$$2\pi a\bar{E}_\theta = -\frac{\pi a^2 p}{c}\bar{H}_z, \quad \text{when } r = a,$$

or

$$\bar{E}_\theta = -\frac{ap}{2c}\bar{H}_z, \quad \text{when } r = a, \tag{5}$$

and this is the required boundary condition.

The general solution of (1) is

$$\bar{H}_z = AI_0(qr)+BK_0(qr),$$

and substituting in (4) and (5) we have for $A$ and $B$

$$\left.\begin{aligned} AI_0(qb)+BK_0(qb) &= H_0/p, \\ A[hI_0'(qa)-qI_0(qa)]+B[hK_0'(qa)-qK_0(qa)] &= 0, \end{aligned}\right\}$$

where $h = 2\mu/a$.

Solving, we find

$$\bar{H}_z = \frac{H_0}{p}\frac{I_0(qr)[hK_0'(qa)-qK_0(qa)]-K_0(qr)[hI_0'(qa)-qI_0(qa)]}{I_0(qb)[hK_0'(qa)-qK_0(qa)]-K_0(qb)[hI_0'(qa)-qI_0(qa)]}. \tag{6}$$

Thus, using the Inversion Theorem,

$$H_z = \frac{H_0}{2i\pi}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{I_0(\nu r)[hK_0'(\nu a)-\nu K_0(\nu a)]-K_0(\nu r)[hI_0'(\nu a)-\nu I_0(\nu a)]}{I_0(\nu b)[hK_0'(\nu a)-\nu K_0(\nu a)]-K_0(\nu b)[hI_0'(\nu a)-\nu I_0(\nu a)]}\frac{d\lambda}{\lambda}, \tag{7}$$

where $\nu = \sqrt{(\lambda/k)}$.

The integrand of (7) is a single-valued function of $\lambda$ with a pole at $\lambda = 0$. To find its other poles put $\lambda = -k\alpha^2$ in the other factor of the denominator, which becomes†

$$\begin{aligned} D(\alpha) &= I_0(ib\alpha)[hK_0'(ia\alpha)-i\alpha K_0(ia\alpha)]-K_0(ib\alpha)[hI_0'(ia\alpha)-i\alpha I_0(ia\alpha)] \\ &= -\tfrac{1}{2}i\pi\{J_0(b\alpha)[hY_1(a\alpha)-\alpha Y_0(a\alpha)]-Y_0(b\alpha)[hJ_1(a\alpha)-\alpha J_0(a\alpha)]\}. \end{aligned} \tag{8}$$

It can be shown that the zeros of (8) are all real and simple. So if these are $\pm\alpha_1$, $\pm\alpha_2$,..., the poles of the integrand of (7) are $\lambda = 0$ and $\lambda = -k\alpha_s^2$, $s = 1, 2,\ldots$.

† $I_0(iz) = J_0(z)$, $K_0(iz) = -\frac{1}{2}\pi i\{J_0(z)-iY_0(z)\}$, $J_0'(z) = -J_1(z)$, $Y_0'(z) = -Y_1(z)$.

Then using the contour of Fig. 10 it follows in the usual way that the line integral of (7) equals $2i\pi$ times the sum of the residues at the poles of its integrand.

The residue† at $\lambda = 0$ is 1.

Using (8), we see that the residue at $\lambda = -k\alpha_s^2$ is

$$-i\pi e^{-k\alpha_s^2 t}\times$$
$$\times\frac{\{J_0(r\alpha_s)[hY_1(a\alpha_s)-\alpha_s Y_0(a\alpha_s)]-Y_0(r\alpha_s)[hJ_1(a\alpha_s)-\alpha_s J_0(a\alpha_s)]\}}{\alpha_s[dD(\alpha)/d\alpha]_{\alpha=\alpha_s}}. \tag{9}$$

Now from (8)

$$\frac{2i}{\pi}\frac{dD(\alpha)}{d\alpha}$$
$$= -bJ_1(b\alpha)[hY_1(a\alpha)-\alpha Y_0(a\alpha)]+bY_1(b\alpha)[hJ_1(a\alpha)-\alpha J_0(a\alpha)]+$$
$$+\frac{1}{\alpha}J_0(b\alpha)\{\alpha(ah-1)Y_0(a\alpha)-(h-a\alpha^2)Y_1(a\alpha)\}-$$
$$-\frac{1}{\alpha}Y_0(b\alpha)\{\alpha(ah-1)J_0(a\alpha)-(h-a\alpha^2)J_1(a\alpha)\}, \tag{10}$$

where we have used the recurrence formulae

$$\left.\begin{aligned} zJ_n'(z) &= zJ_{n-1}(z)-nJ_n(z), \\ zY_n'(z) &= zY_{n-1}(z)-nY_n(z). \end{aligned}\right\} \tag{11}$$

When $\alpha = \alpha_s$, a zero of (8), we have

$$\frac{J_0(b\alpha_s)}{hJ_1(a\alpha_s)-\alpha_s J_0(a\alpha_s)} = \frac{Y_0(b\alpha_s)}{hY_1(a\alpha_s)-\alpha_s Y_0(a\alpha_s)} = \tau, \quad \text{say.} \tag{12}$$

Introducing this in (10) and using the result

$$J_0(z)Y_1(z)-Y_0(z)J_1(z) = -\frac{2}{\pi z}, \tag{13}$$

we get

$$\frac{2i}{\pi}\left[\frac{dD(\alpha)}{d\alpha}\right]_{\alpha=\alpha_s} = \frac{2}{\pi\alpha_s}\left\{\tau(h^2+\alpha_s^2-2h/a)-\frac{1}{\tau}\right\}$$
$$= \frac{2\{(h^2+\alpha_s^2-2h/a)J_0^2(b\alpha_s)-[hJ_1(a\alpha_s)-\alpha_s J_0(a\alpha_s)]^2\}}{\pi\alpha_s J_0(b\alpha_s)[hJ_1(a\alpha_s)-\alpha_s J_0(a\alpha_s)]}. \tag{14}$$

† $I_0(z) = 1+\frac{1}{4}z^2+\dots$, $K_0(z) = -\log\frac{1}{2}z+\dots$.

Thus, finally, we obtain

$$H_z = H_0 + \pi H_0 \sum_{s=1}^{\infty} e^{-k\alpha_s^2 t} \frac{[hJ_1(a\alpha_s)-\alpha_s J_0(a\alpha_s)]^2\{J_0(r\alpha_s)Y_0(b\alpha_s)-Y_0(r\alpha_s)J_0(b\alpha_s)\}}{(h^2+\alpha_s^2-2h/a)J_0^2(b\alpha_s)-[hJ_1(a\alpha_s)-\alpha_s J_0(a\alpha_s)]^2}.$$

**100.** *An infinite conducting circular cylinder of radius $a$ is free from electric and magnetic fields. At $t = 0$ a magnetic field which is uniform at a large distance from the cylinder is established perpendicular to the axis of the cylinder. The disturbance in the region outside the cylinder is supposed to be propagated infinitely rapidly.*

The $z$-axis is chosen along the axis of the cylinder, and the $x$-axis in the direction of the applied field $H_0$.

The field outside the cylinder is given by

$$H_r = -\frac{\partial\phi}{\partial r}, \qquad H_\theta = -\frac{\partial\phi}{r\,\partial\theta},$$

where
$$\nabla^2\phi = 0. \tag{1}$$

Also $H_r$ and $H_\theta$ are to be chosen so that

$$H_r \to H_0\cos\theta, \qquad H_\theta \to -H_0\sin\theta, \quad \text{as } r\to\infty. \tag{2}$$

A solution† of (1) satisfying (2), and with $\phi$ proportional to $\cos\theta$, is

$$\left.\begin{aligned} \phi &= \left(-H_0 r+\frac{A}{r}\right)\cos\theta, \\ H_r &= \left(H_0+\frac{A}{r^2}\right)\cos\theta, \\ H_\theta &= \left(-H_0+\frac{A}{r^2}\right)\sin\theta, \end{aligned}\right\} \tag{3}$$

where $A$, a function of $t$, is to be determined from the boundary conditions at $r = a$.

Within the cylinder we have to satisfy § 97 (13), (14), (15),

† Throughout this chapter, as in the corresponding static problems, we merely seek a solution satisfying the prescribed conditions. The question of uniqueness will not be discussed.

with $H_r^{(0)} = H_\theta^{(0)} = E_z^{(0)} = 0$ and $q^2 = 4\pi\sigma\mu p/c^2$. Then § 97 (15) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{E}_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \bar{E}_z}{\partial \theta^2}-q^2\bar{E}_z = 0.$$

Here let $\bar{E}_z = X(r)\sin\theta$. Then $X$ must satisfy

$$\frac{d^2X}{dr^2}+\frac{1}{r}\frac{dX}{dr}-\left(q^2+\frac{1}{r^2}\right)X \quad 0.$$

Therefore, choosing the solution which is finite at $r = 0$,

$$X = BI_1(qr), \quad \text{and} \quad \bar{E}_z = BI_1(qr)\sin\theta. \tag{4}$$

Hence, using § 97 (13) and (14),

$$\left.\begin{aligned}\bar{H}_r &= -\frac{c}{\mu p}\frac{\partial \bar{E}_z}{r\,\partial\theta} = -\frac{c}{\mu p r}BI_1(qr)\cos\theta,\\ \bar{H}_\theta &= \frac{c}{\mu p}\frac{\partial \bar{E}_z}{\partial r} = \frac{qc}{\mu p}BI_1'(qr)\sin\theta.\end{aligned}\right\} \quad r < a. \tag{5}$$

Also, from (3),

$$\left.\begin{aligned}\bar{H}_r &= \left(\frac{H_0}{p}+\frac{\bar{A}}{r^2}\right)\cos\theta,\\ \bar{H}_\theta &= \left(-\frac{H_0}{p}+\frac{\bar{A}}{r^2}\right)\sin\theta.\end{aligned}\right\} \quad r > a. \tag{6}$$

The boundary conditions at $r = a$ are continuity of tangential magnetic force and normal magnetic induction. These require

$$\left.\begin{aligned}\frac{qc}{\mu p}BI_1'(qa) &= -\frac{H_0}{p}+\frac{\bar{A}}{a^2},\\ -\frac{c}{ap}BI_1(qa) &= \frac{H_0}{p}+\frac{\bar{A}}{a^2}.\end{aligned}\right\}$$

Solving, we have

$$\left.\begin{aligned}B &= -\frac{2\mu a H_0}{c[aqI_1'(qa)+\mu I_1(qa)]},\\ \bar{A} &= -\frac{a^2H_0}{p}\frac{qaI_1'(qa)-\mu I_1(qa)}{qaI_1'(qa)+\mu I_1(qa)}.\end{aligned}\right\} \tag{7}$$

Substituting these values in (5) and (6) gives the transforms

of $H_r$ and $H_\theta$ inside and outside the cylinder. We consider only the interior of the cylinder, then, from (7) and (5),

$$\bar{H}_r = \frac{2aH_0}{pr} \frac{I_1(qr)}{[aqI_1'(qa)+\mu I_1(qa)]}\cos\theta, \quad r < a, \tag{8}$$

$$\bar{H}_\theta = -\frac{2qaH_0}{p} \frac{I_1'(qr)}{[aqI_1'(qa)+\mu I_1(qa)]}\sin\theta, \quad r < a. \tag{9}$$

From (8), by the Inversion Theorem,

$$H_r = \frac{aH_0\cos\theta}{\pi i r}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda} \frac{I_1\{r\sqrt{(\lambda/k)}\}}{a\sqrt{(\lambda/k)}I_1'\{a\sqrt{(\lambda/k)}\}+\mu I_1\{a\sqrt{(\lambda/k)}\}}\,d\lambda, \tag{10}$$

where $k = c^2/4\pi\sigma\mu$.

The integrand of (10) is a single-valued function of $\lambda$. Its poles are $\lambda = 0$, and $\lambda = -(k/a^2)\alpha_s^2$, $s = 1, 2,\ldots$, where the $\alpha_s$ are the roots (all real and simple) of †

$$zJ_0(z)+(\mu-1)J_1(z) = 0.$$

Since

$$\left[\frac{d}{dz}[zJ_0(z)+(\mu-1)J_1(z)]\right]_{z=\alpha_s} = J_0(\alpha_s)\frac{(\alpha_s^2+\mu^2-1)}{\mu-1},$$

the residue of the integrand of (10) at $\lambda = -k\alpha_s^2/a^2$ is

$$\frac{2(\mu-1)}{\alpha_s(\alpha_s^2+\mu^2-1)} \frac{J_1(r\alpha_s/a)}{J_0(\alpha_s)} e^{-k\alpha_s^2 t/a^2}.$$

Also, since $I_1(z) = z+\frac{1}{8}z^3+\ldots$, the residue at $\lambda = 0$ is

$$\frac{r}{a(\mu+1)}.$$

So, finally,

$$H_r = \frac{2H_0\cos\theta}{\mu+1} + \sum_{s=1}^{\infty} \frac{4a(\mu-1)H_0\cos\theta}{r\alpha_s(\alpha_s^2+\mu^2-1)} \frac{J_1(r\alpha_s/a)}{J_0(\alpha_s)} e^{-k\alpha_s^2 t/a^2}.$$

Similarly,

$$H_\theta = -\frac{2H_0\sin\theta}{\mu+1} - \sum_{s=1}^{\infty} \frac{4(\mu-1)H_0\sin\theta}{(\alpha_s^2+\mu^2-1)} \frac{J_1'(r\alpha_s/a)}{J_0(\alpha_s)} e^{-k\alpha_s^2 t/a^2}.$$

† It is supposed that $\mu > 1$. If $\mu = 1$, the denominator of (10) reduces by the recurrence formula $zI_1'(z)+I_1(z) = zI_0(z)$ to

$$a\sqrt{(\lambda/k)}I_0\{a\sqrt{(\lambda/k)}\}.$$

A problem with $\mu = 1$ is discussed in § 101.

The problem of a conducting sphere placed in a uniform magnetic field may be treated in a similar way.

**101.** *The problem of* § 100 *with an oscillating field* $H_0 \sin \omega t$ *in place of the constant field* $H_0$.

Here $H_0/p$ is to be replaced by $\omega H_0/(p^2+\omega^2)$ in § 100 (10), giving

$$H_r = \frac{a\omega H_0 \cos\theta}{i\pi r} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda^2+\omega^2} \frac{I_1(\nu r)}{a\nu I_1'(\nu a)+\mu I_1(\nu a)}\, d\lambda, \tag{1}$$

where $\nu = \sqrt{(\lambda/k)}$.

Here we consider only the case $\mu = 1$ (see footnote on p. 221) in which (1) becomes

$$H_r = \frac{a\omega H_0 \cos\theta}{\pi i r} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda^2+\omega^2} \frac{I_1(\nu r)}{a\nu I_0(\nu a)}\, d\lambda. \tag{2}$$

The poles of the integrand of (2) are

$$\lambda = \pm i\omega \quad \text{and} \quad \lambda = -k\alpha_s^2/a^2,$$

where $\pm\alpha_s$, $s = 1, 2, 3,\ldots$, are the roots (all real and simple) of

$$J_0(z) = 0.$$

The residue of the integrand of (2) at $\lambda = i\omega$ is,† writing $\omega' = \sqrt{(\omega/k)}$,

$$\frac{e^{i\omega t}}{2i\omega} \frac{I_1(r\omega' i^{\frac{1}{2}})}{a\omega' i^{\frac{1}{2}} I_0(r\omega' i^{\frac{1}{2}})} = -\frac{e^{i\omega t}}{2\omega\omega' a i^{\frac{1}{2}}} \frac{\operatorname{ber}_1 r\omega' + i \operatorname{bei}_1 r\omega'}{\operatorname{ber} a\omega' + i \operatorname{bei} a\omega'}.$$

The residue at $\lambda = -k\alpha_s^2/a^2$ is

$$\frac{2ka^2}{k^2\alpha_s^4+\omega^2 a^4} \frac{J_1(r\alpha_s/a)}{J_1(\alpha_s)} e^{-k\alpha_s^2 t/a^2}.$$

Therefore,

$$H_r = -\frac{2H_0\cos\theta}{r\omega'} \left[\frac{\operatorname{ber}_1^2 \omega' r + \operatorname{bei}_1^2 \omega' r}{\operatorname{ber}^2 \omega' a + \operatorname{bei}^2 \omega' a}\right]^{\frac{1}{2}} \cos[\omega t - \tfrac{1}{4}\pi + \delta_1 - \delta_0] +$$

$$+ \frac{4a^3 k\omega H_0 \cos\theta}{r} \sum_{s=1}^{\infty} \frac{1}{k^2\alpha_s^4+\omega^2 a^4} \frac{J_1(r\alpha_s/a)}{J_1(\alpha_s)} e^{-k\alpha_s^2 t/a^2},$$

where $\tan\delta_1 = \dfrac{\operatorname{bei}_1 r\omega'}{\operatorname{ber}_1 r\omega'}$, $\tan\delta_0 = \dfrac{\operatorname{bei} a\omega'}{\operatorname{ber} a\omega'}$.

† $\operatorname{ber}_n x + i \operatorname{bei}_n x = e^{\frac{1}{2}n\pi i} I_n(x e^{\frac{1}{4}\pi i})$.

**102.** *Charge is bound with surface density $D\cos\theta$ on the surface of a perfectly conducting sphere of radius $a$. At $t=0$ the charge is released. To find the disturbance in the medium ($\kappa=\mu=1$, $\sigma=0$) outside the sphere.*†

The initial conditions are the electrostatic field of the charge distribution $D\cos\theta$, namely,

$$E_r^{(0)}=\frac{2A\cos\theta}{r^3}, \qquad E_\theta^{(0)}=\frac{A\sin\theta}{r^3}, \qquad H_\phi^{(0)}=0,$$

where $A=2\pi Da^3$.

The problem is independent of $\phi$; so we use the subsidiary equations (22), (23), (25) of § 97. These become

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\bar{H}_\phi]=\frac{p}{c}\bar{E}_r-\frac{2A\cos\theta}{cr^3} \tag{1}$$

$$-\frac{1}{r}\frac{\partial(r\bar{H}_\phi)}{\partial r}=\frac{p}{c}\bar{E}_\theta-\frac{A\sin\theta}{cr^3}, \tag{2}$$

$$\frac{1}{r}\frac{\partial^2(r\bar{H}_\phi)}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\bar{H}_\phi)\right]-\frac{p^2}{c^2}\bar{H}_\phi=0. \tag{3}$$

These have to be solved with boundary condition $\bar{E}_\theta=0$, when $r=a$, for every $\theta$. Because of the occurrence of $\sin\theta$ in the right-hand side of (2) we seek a solution of (3) of the form $\bar{H}_\phi=R(r)\sin\theta$. Then (3) gives for $R$

$$\frac{1}{r}\frac{d^2(rR)}{dr^2}-\left(\frac{2}{r^2}+\frac{p^2}{c^2}\right)R=0. \tag{4}$$

To solve (4) put $R=r^{-\frac{1}{2}}Y$. Then $Y$ satisfies

$$\frac{d^2Y}{dr^2}+\frac{1}{r}\frac{dY}{dr}-\left(\frac{9}{4r^2}+\frac{p^2}{c^2}\right)Y=0.$$

The solution of this, which remains finite as $r\to\infty$, is $Y=BK_{\frac{3}{2}}(pr/c)$. Thus

$$\bar{H}_\phi=Br^{-\frac{1}{2}}K_{\frac{3}{2}}(pr/c)\sin\theta. \tag{5}$$

The boundary condition $\bar{E}_\theta=0$ at $r=a$ now becomes, using (2),

$$\frac{B}{a}\left[\frac{d}{dr}\left\{r^{\frac{1}{2}}K_{\frac{3}{2}}\left(\frac{pr}{c}\right)\right\}\right]_{r=a}=\frac{A}{ca^3}.$$

† Love, *Proc. Lond. Math. Soc.* (2), 2 (1904), 102.

Hence, using the result

$$K_{\frac{3}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left(1+\frac{1}{z}\right),$$

we find
$$B = -\left(\frac{2p}{\pi c}\right)^{\frac{1}{2}} \frac{Ape^{pa/c}}{p^2a^2+pac+c^2}.$$

Therefore,
$$\bar{H}_\phi = -A\frac{(pr+c)\sin\theta}{r^2(p^2a^2+pac+c^2)}e^{-p(r-a)/c}, \tag{6}$$

and from (1) and (2)

$$\bar{E}_r = \frac{2A\cos\theta}{pr^3} - \frac{2cA(pr+c)\cos\theta}{pr^3(p^2a^2+pac+c^2)}e^{-p(r-a)/c}, \tag{7}$$

$$\bar{E}_\theta = \frac{A\sin\theta}{pr^3} - \frac{A(p^2r^2+prc+c^2)}{pr^3(p^2a^2+pac+c^2)}\sin\theta\, e^{-p(r-a)/c}. \tag{8}$$

From (6) we have

$$\bar{H}_\phi = -\frac{A}{a^2r}\sin\theta\frac{(p+c/2a)+(c/r-c/2a)}{(p+c/2a)^2+3c^2/4a^2}e^{-p(r-a)/c},$$

and thus, by Theorems IV and V of § 3,

$$H_\phi = -\frac{A\sin\theta}{a^2r}\left\{\cos\frac{c\sqrt{3}}{2a}\left[t-\frac{r-a}{c}\right]+\right.$$
$$\left.+\frac{(2a-r)}{r\sqrt{3}}\sin\frac{c\sqrt{3}}{2a}\left[t-\frac{r-a}{c}\right]\right\}e^{-(c/2a)[t-(r-a)/c]}, \quad \text{when } t > \frac{r-a}{c}$$
$$= 0, \quad \text{when } t < \frac{r-a}{c} \tag{9}$$

Putting $\vartheta = (c/2a)[t-(r-a)/c]$ and $\tan\delta = (r-2a)/r\sqrt{3}$, (9) becomes

$$H_\phi = -\frac{\sin\theta}{r}\frac{2A}{a^2\sqrt{3}}\left(1-\frac{a}{r}+\frac{a^2}{r^2}\right)^{\frac{1}{2}}e^{-\vartheta}\cos(\vartheta\sqrt{3}+\delta), \quad \vartheta > 0,$$
$$= 0, \quad \vartheta < 0.$$

In the same way $E_r$ and $E_\theta$ may be obtained from (7) and (8). The charge distribution on the sphere is obtained from

$$\frac{1}{4\pi}(\bar{E}_r)_{r=a} = \frac{A\cos\theta}{2\pi pa^3} - \frac{cA(pa+c)\cos\theta}{2\pi pa^3(p^2a^2+pac+c^2)}$$
$$= \frac{Ap\cos\theta}{2\pi a^3\{(p+c/2a)^2+3c^2/4a^2\}}.$$

Thus

$$\frac{1}{4\pi}(E_r)_{r=a} = \frac{A}{2\pi a^3}e^{-ct/(2a)}\left[\cos\frac{ct}{2a}\sqrt{3} - \frac{1}{\sqrt{3}}\sin\frac{ct}{2a}\sqrt{3}\right]\cos\theta.$$

The case of a sphere of finite conductivity may be treated along the same lines. Equation (5) will still hold for the space outside the sphere, and proceeding in the same way we should find inside the sphere

$$\bar{H}_\phi = B'r^{-\frac{1}{2}}I_{\frac{3}{2}}(qr)\sin\theta,$$

where $q = \sqrt{(4\pi\sigma\mu p/c^2)}$. The constants $B$ and $B'$ are then found from the boundary conditions, continuity of $H_\phi$ and $E_\theta$, at $r = a$.

**103.** *The problem of* § 102 *with the initial surface density a zonal harmonic of order* 2, $P_2(\cos\theta)$.

Here the initial conditions are the electrostatic field of this distribution, namely,

$$E_r^{(0)} = \frac{3A}{2r^4}(3\cos^2\theta - 1), \qquad E_\theta^{(0)} \quad \frac{3A\sin\theta\cos\theta}{r^4}, \qquad H_\phi^{(0)} = 0,$$

where $A = \frac{4}{5}\pi a^4$.

Equations (22), (23), (25) of § 97 then become

$$\overline{r\sin\theta\,\partial\theta} \qquad = \frac{p}{c}\bar{E}_r - \frac{3A}{2cr^4}(3\cos^2\theta - 1), \tag{1}$$

$$1\ \partial(r. \quad \overline{\partial r} \qquad = \frac{p}{c}\bar{E}_\theta - \frac{3A}{cr^4}\sin\theta\cos\theta, \tag{2}$$

$$\frac{1}{r}\frac{\partial^2(r\bar{H}_\phi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin \qquad -\frac{p^2}{c^2}\bar{H}_\phi = 0. \tag{3}$$

We seek a solution of (3) of type $\bar{H}_\phi = R(r)\sin\theta\cos\theta$. Then $R$ has to satisfy

$$\frac{1}{r}\frac{d^2(rR)}{dr^2} - \left(\frac{6}{r^2} + \frac{p^2}{c^2}\right)R = 0.$$

Putting $R \quad r^{-\frac{1}{2}}Y$ in this, $Y$ has to satisfy

$$\frac{d^2Y}{dr^2} + \frac{1}{r}\frac{dY}{dr} - \left(\frac{25}{4r^2} + \frac{p^2}{c^2}\right)Y = 0.$$

The solution of this, finite as $r \to \infty$, is $K_{\frac{5}{2}}(pr/c)$, and so

$$\bar{H}_\phi = Br^{-\frac{1}{2}}K_{\frac{5}{2}}(pr/c)\sin\theta\cos\theta. \tag{4}$$

As in § 102 the boundary condition is $\bar{E}_\theta = 0$, when $r = a$, for every $\theta$.

Using (2), this requires

$$\frac{B}{a}\left[\frac{d}{dr}\left\{r^{\frac{1}{2}}K_{\frac{3}{2}}\left(\frac{pr}{c}\right)\right\}\right]_{r=a} = \frac{3A}{ca^4}.$$

Thus, using the result

$$K_{\frac{3}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left(1+\frac{3}{z}+\frac{3}{z^2}\right),$$

we have

$$B = -\left(\frac{2p}{\pi c}\right)^{\frac{1}{2}} e^{pa/c}\frac{3Ap^2}{p^3a^3+3cp^2a^2+6c^2pa+6c^3},$$

and

$$\bar{H}_\phi = -\frac{3A}{r^3}\frac{p^2r^2+3cpr+3c^2}{p^3a^3+3cp^2a^2+6c^2pa+6c^3}e^{-p(r-a)/c}\sin\theta\cos\theta. \quad (5)$$

$\bar{E}_r$ and $\bar{E}_\theta$ may now be determined from (1) and (2).

The equation

$$p^3+\frac{3c}{a}p^2+\frac{6c^2}{a^2}p+\frac{6c^3}{a^3} = 0$$

has one negative real root and two complex roots with negative real parts.† Denoting these by $\lambda_1$, $\lambda_2$, and $\lambda_3$ respectively, we have from (5)

$$\begin{aligned} H_\phi = &-\frac{3A}{a^3r^3}\frac{\lambda_1^2 r^2+3c\lambda_1 r+3c^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}e^{\lambda_1[t-(r-a)/c]}\sin\theta\cos\theta- \\ &-\frac{3A}{a^3r^3}\frac{\lambda_2^2 r^2+3c\lambda_2 r+3c^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}e^{\lambda_2[t-(r-a)/c]}\sin\theta\cos\theta- \\ &-\frac{3A}{a^3r^3}\frac{\lambda_3^2 r^2+3c\lambda_3 r+3c^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}e^{\lambda_3[t-(r-a)/c]}\sin\theta\cos\theta, \end{aligned}$$

when $t > (r-a)/c$ and $H_\phi = 0$, when $t < (r-a)/c$.

The first term is of exponential type, and the other two represent damped harmonic vibrations.

The case in which the initial charge distribution is a zonal harmonic of order $n$ may be treated in the same way. In this case the equation of type (4) for $\bar{H}_\phi$ will involve $K_{n+\frac{1}{2}}(pr/c)$.

† The roots are approximately $-1{\cdot}60c/a$, $(-0{\cdot}70\pm 1{\cdot}81i)c/a$.

**104.** *The field due to an oscillating electric dipole along the z-axis at the origin.*

Suppose that at $t = 0$, when the field is that of a dipole of moment $M_0$, the moment starts to oscillate like $M_0 \cos\omega t$. To find the disturbance in the external medium in which $\kappa = \mu = 1$, $\sigma = 0$.

The dipole is taken to consist of charges $\pm Q$ whose distances from the origin are $z = \pm l\cos\omega t$, for $t \geqslant 0$, so that $M_0 = 2Ql$. The magnetic field of the system at points $(r,\theta)$, so near the origin that the time of transit of disturbances to them may be neglected, is†

$$H_\phi = -\frac{2Q}{cr^2}\omega l \sin\omega t \sin\theta = -\frac{\omega M_0}{cr^2}\sin\omega t \sin\theta.$$

Thus the solution for $r > 0$ must satisfy

$$H_\phi \to -\frac{\omega M_0}{cr^2}\sin\omega t\sin\theta, \quad \text{as } r \to 0,$$

i.e.
$$\bar{H}_\phi \to -\frac{\omega^2 M_0}{cr^2(p^2+\omega^2)}\sin\theta, \quad \text{as } r \to 0 \tag{1}$$

The initial conditions are the electrostatic field of a dipole of moment $M_0$, namely,

$$E_r^{(0)} = 2M_0\cos\theta \qquad E_\theta^{(0)} = \frac{M_0\sin\theta}{r^3} \qquad H_\phi^{(0)} = 0.$$

Thus the subsidiary equations § 97 (22), (23), (25) become

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\bar{H}_\phi] = \frac{p}{c}\bar{E}_r - \frac{2M_0\cos\theta}{cr^3}, \tag{2}$$

$$-\frac{1}{r}\frac{\partial(r\bar{H}_\phi)}{\partial r} = \frac{p}{c}\bar{E}_\theta - \frac{M_0\sin\theta}{cr^3}, \tag{3}$$

$$\frac{1}{r}\frac{\partial^2(r\bar{H}_\phi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\bar{H}_\phi)\right] - \frac{p^2}{c^2}\bar{H}_\phi = 0. \tag{4}$$

As in § 102, a solution of (4) with $\bar{H}_\phi \propto \sin\theta$ is

$$\bar{H}_\phi = Br^{-\frac{1}{2}}K_{\frac{3}{2}}(pr/c)\sin\theta = \frac{B}{r}\left(\frac{\pi c}{2p}\right)^{\frac{1}{2}}\left(1+\frac{c}{rp}\right)e^{-rp/c}\sin\theta, \tag{5}$$

† Cf. Jeans, *Electricity and Magnetism* (5th ed., 1925), § 572.

where $B$ is to be determined from the boundary condition (1). This gives

$$B = -\frac{M_0\,\omega^2 p}{c^2(p^2+\omega^2)}\left(\frac{2p}{\pi c}\right)^{\frac{1}{2}}.$$

Using this value we have, from (5), (2), and (4),

$$\bar{H}_\phi = -\frac{M_0\,\omega^2(rp+c)}{r^2c^2(p^2+\omega^2)}e^{-pr/c}\sin\theta, \tag{6}$$

$$\bar{E}_r = \frac{2M_0\cos\theta}{pr^3} - \frac{2M_0\,\omega^2(rp+c)}{pcr^3(p^2+\omega^2)}e^{-pr/c}\cos\theta, \tag{7}$$

$$\bar{E}_\theta = \frac{M_0\sin\theta}{pr^3} - \frac{M_0\,\omega^2(p^2r^2+prc+c^2)}{pc^2r^3(p^2+\omega^2)}e^{-pr/c}\sin\theta. \tag{8}$$

Hence, using § 3, Theorem V, we obtain

$$H_\phi = -\frac{M_0\,\omega^2}{r^2c^2}\left[r\cos\omega\left(t-\frac{r}{c}\right)+\frac{c}{\omega}\sin\omega\left(t-\frac{r}{c}\right)\right]\sin\theta,$$
when $t > r/c$,

$= 0,\quad$ when $t < r/c$,

$$E_r = -\frac{2M_0\,\omega}{cr^3}\left[r\sin\omega\left(t-\frac{r}{c}\right)-\frac{c}{\omega}\cos\omega\left(t-\frac{r}{c}\right)\right]\cos\theta,$$
when $t > r/c$,

$= \dfrac{2M_0\cos\theta}{r^3},\quad$ when $t < r/c$,

$$E_\theta = -\frac{M_0\,\omega^2}{c^2r^3}\left[\left(r^2-\frac{c^2}{\omega^2}\right)\cos\omega\left(t-\frac{r}{c}\right)+\frac{rc}{\omega}\sin\omega\left(t-\frac{r}{c}\right)\right]\sin\theta,$$
when $t > r/c$,

$= \dfrac{M_0\sin\theta}{r^3},\quad$ when $t < r/c$.

**105.** *The retarded potential formulae.*

It is required to solve the equation

$$-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \rho(x,y,z,t) \tag{1}$$

in the region

$$t > 0,\quad -\infty < x < \infty,\quad -\infty < y < \infty,\quad -\infty < z < \infty:$$

$\rho(x,y,z,t)$ is a known function; $\phi$ and $\partial\phi/\partial t$ are to take the values $f(x,y,z)$ and $g(x,y,z)$ respectively, when $t = 0$.

The subsidiary equation for (1) with these initial conditions is

$$\nabla^2\bar{\phi}-q^2\bar{\phi} = \bar{\rho}(x,y,z,p)-\frac{1}{c^2}[pf(x,y,z)+g(x,y,z)], \tag{2}$$

where $q^2 = p^2/c^2$.

By Green's theorem† the solution of (2) is

$$\begin{aligned} 4\pi\bar{\phi}(x',y',z') &= -\iint \bar{\phi}(x,y,z)\frac{\partial}{\partial n}\left[\frac{e^{-qR}}{R}\right]dS+\iint \frac{\partial\bar{\phi}}{\partial n}\frac{e^{-qR}}{R}\,dS- \\ &\quad -\iiint \frac{e^{-qR}}{R}\left\{\bar{\rho}(x,y,z,p)-\frac{1}{c^2}[pf(x,y,z)+g(x,y,z)]\right\}dxdydz, \end{aligned} \tag{3}$$

where the integrals are taken over the surface and volume of a closed surface $S$ which is to be made indefinitely large, $\partial/\partial n$ represents a differentiation along the outward normal of this surface, and

$$R^2 = (x-x')^2+(y-y')^2+(z-z')^2.$$

We assume that $\phi$ is such that the surface integrals in (3) vanish as the least diameter of $S$ approaches infinity. Then

$$4\pi\bar{\phi}(x',y',z') = -\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{e^{-qR}}{R}\left[\bar{\rho}-\frac{1}{c^2}(pf+g)\right]dxdydz.$$

Here we may take $x' = y' = z' = 0$ without loss of generality and obtain

$$4\pi\bar{\phi} = 4\pi\bar{\phi}(0,0,0) = -\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{e^{-qr}}{r}\left[\bar{\rho}-\frac{1}{c^2}(pf+g)\right]dxdydz. \tag{4}$$

We transform this to spherical polar coordinates and write

$$\left.\begin{aligned} F(r) &= r\iint f(r,\theta,\psi)\,d\omega, \\ G(r) &= r\iint g(r,\theta,\psi)\,d\omega, \end{aligned}\right\} \tag{5}$$

where $\iint d\omega$ is written throughout for $\int\limits_0^{\pi}\int\limits_0^{2\pi}\sin\theta\,d\theta d\psi$.

† Bateman, *Partial Differential Equations of Mathematical Physics* (1932), p. 189.

Then (4) becomes

$$4\pi\bar{\phi} = -\iint d\omega \int_0^\infty re^{-pr/c}\bar{\rho}(r,\theta,\psi,p)\,dr + \\ +\frac{p}{c^2}\int_0^\infty e^{-pr/c}F(r)\,dr + \frac{1}{c^2}\int_0^\infty e^{-pr/c}G(r)\,dr. \quad (6)$$

To determine $\phi$ from (6) we consider the three terms separately.

By § 3, Theorem V, the function whose transform is $e^{-pr/c}\bar{\rho}(r,\theta,\psi,p)$ is

$$\left.\begin{array}{ll} \rho\left(r,\theta,\psi,t-\frac{r}{c}\right), & \text{when } t > \frac{r}{c}, \\ 0, & \text{when } t < \frac{r}{c}. \end{array}\right\}$$

Thus (assuming that the orders of integration can be interchanged) the first term of (6) makes a contribution to $4\pi\phi$ of

$$-\iint d\omega \int_0^{ct} r\,dr\,\rho\left(r,\theta,\psi,t-\frac{r}{c}\right). \quad (7)$$

Applying the Inversion Theorem to the last term of (6) gives

$$\frac{1}{2\pi ic^2}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\,d\lambda \int e^{-\lambda r/c}G(r)\,dr = \frac{1}{2\pi ic}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\,d\lambda \int e^{-\lambda u}G(cu)\,du$$

$$= \frac{1}{c}G(ct), \quad (8)$$

by Fourier's theorem, equation (12) below.

Applying the Inversion Theorem to the second term of (6) gives

$$\frac{1}{2\pi ic^2}\int_{\gamma-i\infty}^{\gamma+i\infty} \lambda e^{\lambda t}\,d\lambda \int^\infty e^{-\lambda r/c}F(r)\,dr$$

$$= \frac{1}{2\pi ic}\int^{\gamma+i\infty} \lambda e^{\lambda t}\,d\lambda \int^\infty e^{-\lambda u}F(cu)\,du$$

$$= \frac{1}{c}\frac{d}{dt}F(ct) = F'(ct), \quad (9)$$

by equation (13) below; the condition $F(0) = 0$ will by (5) be satisfied if $\lim_{r\to 0} rf(r,\theta,\psi) = 0$.

Therefore, from (6), (7), (8), (9), we have finally

$$4\pi\phi = \frac{1}{c}G(ct)+F'(ct)-\iint d\omega \int\limits_0^{ct} \rho\left(r,\theta,\psi,t-\frac{r}{c}\right)r\,dr, \quad (10)$$

which is the usual result.†

Fourier's Integral Theorem‡ states that (subject to certain conditions on $\phi$)

$$\phi(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-i\alpha x}\,d\alpha \int\limits_{-\infty}^{\infty} e^{i\alpha t}\phi(t)\,dt. \quad (11)$$

Here let
$$\left.\begin{aligned}\phi(t) &= f(t)e^{-\gamma t}, & t &> 0,\\ &= 0, & t &< 0.\end{aligned}\right\}$$

Then, inserting these values, (11) becomes

$$\begin{aligned}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{(\gamma-i\alpha)x}\,d\alpha \int\limits_0^{\infty} e^{-(\gamma-i\alpha)t}f(t)\,dt &= f(x), & x &> 0,\\ &= 0, & x &< 0.\end{aligned}$$

Putting $\gamma-i\alpha = \lambda$, this becomes

$$\left.\begin{aligned}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda x}\,d\lambda \int\limits_0^{\infty} e^{-\lambda t}f(t)\,dt &= f(x), & x &> 0,\\ &= 0, & x &< 0.\end{aligned}\right\} \quad (12)$$

Also, if $f(t) = \psi'(t)$, $t > 0$, and $\psi(0) = 0$,

$$\int\limits_0^{\infty} e^{-\lambda t}f(t)\,dt = \lambda\int\limits_0^{\infty} e^{-\lambda t}\psi(t)\,dt,$$

and (12) becomes

$$\left.\begin{aligned}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \lambda e^{\lambda x}\,d\lambda \int\limits_0^{\infty} e^{-\lambda t}\psi(t)\,dt &= \psi'(x), & x &> 0,\\ &= 0, & x &< 0.\end{aligned}\right\} \quad (13)$$

† Abraham, *Theorie der Elektrizität* (Leipzig, 1905), **2**, § 7 (39).

‡ Cf. § 42.

# MISCELLANEOUS EXAMPLES INVOLVING PARTIAL DIFFERENTIAL EQUATIONS

1. Obtain the solution $v(x,t)$ of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x > 0,\ t > 0,$$

such that $\quad v(0,t) = 0, \quad t > 0,$

and $\quad v(x,0) = v_0, \quad x > 0.$

[$v = v_0 \operatorname{erf}\{x/2\sqrt{(\kappa t)}\}$.

2. Obtain the solution of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x > 0,\ t > 0,$$

with $\quad v = a\sin\omega t, \quad$ when $x = 0,\ t > 0,$

and $\quad v = 0, \quad$ when $x > 0,\ t = 0.$

$$\left[v = a\left\{e^{-x\sqrt{(\omega/2\kappa)}}\sin\left(\omega t - x\sqrt{\frac{\omega}{2\kappa}}\right) + \frac{\omega}{\pi}\int\limits_0^\infty e^{-\rho t}\frac{\sin x\sqrt{(\rho/\kappa)}}{\rho^2+\omega^2}\,d\rho\right\}.\right.$$

3. Obtain the solution of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x > 0,\ t > 0,$$

with $\quad v = \phi(t), \quad$ when $x = 0,\ t > 0,$

and $\quad v = 0, \quad$ when $x > 0,\ t = 0.$

$$\left[v = \frac{x}{2\sqrt{(\pi\kappa)}}\int\limits_0^t \phi(\tau)\frac{e^{-x^2/4\kappa(t-\tau)}}{(t-\tau)^{\frac{3}{2}}}\,d\tau.\right.$$ [Use § 3, Theorem VI.]

4. Obtain the solution of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x > 0,\ t > 0,$$

with $\quad -\frac{\partial v}{\partial x} + hv = ah\cos\omega t, \quad x = 0,\ t > 0,$

and $\quad v = 0, \quad x > 0,\ t = 0.$

$$\left[v = \frac{ahe^{-x\sqrt{(\omega/2\kappa)}}}{\sqrt{[\{h+\sqrt{(\omega/2\kappa)}\}^2+\omega/2\kappa]}}\cos\{\omega t - x\sqrt{(\omega/2\kappa)} - \alpha\} -\right.$$

$$-\frac{2ah\kappa^2}{\pi}\int\limits_0^\infty e^{-\kappa u^2 t}\frac{u^3}{(\kappa^2u^4+\omega^2)}\frac{(h\sin ux + u\cos ux)}{(h^2+u^2)}\,du,$$

where $\quad \alpha = \tan^{-1}\frac{\sqrt{(\omega/2\kappa)}}{h+\sqrt{(\omega/2\kappa)}}.$

5. Obtain the solution of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad x > 0,\ t > 0,$$

such that
$$-\frac{\partial v}{\partial x}+hv = h\phi(t), \quad x = 0,\ t > 0,$$

and
$$v = 0, \qquad x > 0,\ t = 0.$$

$$\left[v = \frac{2h\kappa}{\pi}\int\limits_0^t \phi(t-\tau)\left\{\int\limits_0^\infty e^{-\kappa u^2\tau}\frac{h\sin ux+u\cos ux}{h^2+u^2}u\,du\right\}d\tau.\right.$$

[Use § 3, Theorem VI.]

6. Obtain the solution of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad 0 < x < l,\ t > 0,$$

with $v(0,t) = v_0$ and $v(l,t) = v_1$, when $t > 0$,

and $v(x,0) = 0$, $0 < x < l$.

$$\left[v = v_0\left\{\frac{l-x}{l}-\frac{2}{\pi}\sum_1^\infty \frac{e^{-\kappa(n^2\pi^2/l^2)t}}{n}\sin\frac{n\pi}{l}x\right\}+\right.$$
$$+v_1\left\{\frac{x}{l}+\frac{2}{\pi}\sum_1^\infty \frac{(-1)^n}{n}e^{-\kappa(n^2\pi^2/l^2)t}\sin\frac{n\pi}{l}x\right\}.$$

7. Obtain the solution of

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad 0 < x < l,\ t > 0,$$

with $v = \phi(t)$, $x = 0,\ t > 0$; $v = 0,\ x = l,\ t > 0$,

and $v = 0$, $0 < x < l,\ t = 0$.

$$\left[v = \frac{2\kappa\pi}{l^2}\sum_1^\infty n\sin\frac{n\pi}{l}x\,e^{-\kappa(n^2\pi^2/l^2)t}\int\limits_0^t \phi(\tau)e^{-\kappa(n^2\pi^2/l^2)\tau}\,d\tau.\right.$$

8. Two semi-infinite solids of different materials are in contact along the plane $x = 0$. The initial temperature in $x < 0$ is a constant $V_1$ and that in $x > 0$ is zero. Taking $K_1$, $\kappa_1$ in $x < 0$ and $K_2$, $\kappa_2$ in $x > 0$, show that the temperature at the time $t > 0$ is given by

$$\left.\begin{aligned} v_1 &= \frac{V_1}{1+\sigma}\left[1+\sigma\,\mathrm{erf}\left(\frac{|x|}{2\sqrt{(\kappa_1 t)}}\right)\right], \quad x < 0 \\ v_2 &= \frac{V_1}{1+\sigma}\left[1-\mathrm{erf}\left(\frac{x}{2\sqrt{(\kappa_2 t)}}\right)\right], \quad x > 0 \end{aligned}\right\}, \text{ where } \sigma = \frac{K_2\sqrt{\kappa_1}}{K_1\sqrt{\kappa_2}}.$$

9. In the solid of Ex. 8, if the initial temperature in $x < 0$ is zero and in $x > 0$ is a constant $V_2$, show that the temperature at $t$ is given by

$$v_1 = \frac{V_2\sigma}{1+\sigma}\left[1-\operatorname{erf}\left(\frac{|x|}{2\sqrt{(\kappa_1 t)}}\right)\right], \quad x < 0,$$

$$v_2 = \frac{V_2\sigma}{1+\sigma}\left[1+\frac{1}{\sigma}\operatorname{erf}\left(\frac{x}{2\sqrt{(\kappa_2 t)}}\right)\right], \quad x > 0,$$

where $\sigma$ is as in Ex. 8.

10. Find the temperature due to a unit instantaneous plane source at $x'$ ($> 0$) at $t = 0$ in the solid of Ex. 8.

[The temperature $v_2$ in $x > 0$ is the same as if the whole solid had been of the $K_2$ material and another source of strength $\dfrac{K_2\sqrt{\kappa_1}-K_1\sqrt{\kappa_2}}{K_2\sqrt{\kappa_1}+K_1\sqrt{\kappa_2}}$ had been placed at $-x'$.

The temperature $v_1$ in $x < 0$ is the same as if the whole solid had been of the $K_1$ material and a source of strength $\dfrac{2K_2\sqrt{\kappa_1}}{K_2\sqrt{\kappa_1}+K_1\sqrt{\kappa_2}}\sqrt{\dfrac{\kappa_1}{\kappa_2}}$ had been placed at $x'\sqrt{(\kappa_1/\kappa_2)}$.

11. Obtain the solution of the equation

$$\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2}, \quad 0 < x < l,\ t > 0,$$

with $\quad v = 0, \quad$ when $x = 0$ and $x = l$, $t > 0$,

and
$$\left.\begin{aligned} v &= x, & 0 < x < \tfrac{1}{2}l,\\ &= l-x, & \tfrac{1}{2}l < x < l. \end{aligned}\right\} t = 0.$$

[Show that the subsidiary equation is

$$\left.\begin{aligned} \frac{d^2\bar{v}}{dx^2}-q^2\bar{v} &= -\frac{x}{\kappa}, & 0 < x < \tfrac{1}{2}l,\\ &= -\frac{l-x}{\kappa}, & \tfrac{1}{2}l < x < l, \end{aligned}\right\} q^2 = p/\kappa$$

with $\quad \bar{v} = 0, \quad$ when $x = 0$ and $x = l$.

Verify that this is satisfied by

$$\bar{v} = \frac{1}{\kappa q^2}\left[x-\frac{1}{q}\frac{\sinh qx}{\cosh\frac{1}{2}ql}\right], \quad 0 < x < \tfrac{1}{2}l,$$

$$= \frac{1}{\kappa q^2}\left[(l-x)-\frac{1}{q}\frac{\sinh q(l-x)}{\cosh\frac{1}{2}ql}\right], \quad \tfrac{1}{2}l < x < l.$$

Hence show that

$$v = \frac{4l}{\pi^2}\sum_1^\infty \frac{(-1)^n}{(2n+1)^2}e^{-\kappa\{(2n+1)^2\pi^2 t\}/l^2}\sin\frac{(2n+1)\pi}{l}x.$$

12. Find the temperature $v$ at time $t$ due to a unit instantaneous line source generated at $t = 0$ along the axis of the cylinder $r = a$, the temperature of the cylinder then being zero and its surface thereafter kept at zero.

$$\left[v = \frac{1}{\pi a^2} \sum_{\alpha} e^{-\kappa\alpha^2 t} \frac{J_0(\alpha r)}{J_1^2(\alpha a)},\right.$$

the summation being taken over the positive roots of $J_0(\alpha a) = 0$.
[Use the integral

$$K_0(qr) = \tfrac{1}{2}\int_0^\infty e^{-pt-r^2/4\kappa t}\frac{dt}{t}, \quad \text{where } q^2 = p/\kappa.$$

13. If in Ex. 12 the surface condition is that no heat escapes across $r = a$, show that

$$v = \frac{1}{\pi a^2}\left[1+\sum_{\alpha} e^{-\kappa\alpha^2 t}\frac{J_0(\alpha r)}{J_0^2(\alpha a)}\right],$$

where the summation is taken over the positive roots of $J_1(\alpha a) = 0$.

14. A solid is bounded internally by the cylinder $r = a$ and extends to infinity. The initial temperature is zero and the surface is kept at a constant temperature $v_0$. The temperature in the solid at $t > 0$ is denoted by $v$. Show that

$$\bar{v} = \frac{v_0}{p}\frac{K_0(qr)}{K_0(qa)}, \quad q^2 = \frac{p}{\kappa},$$

and, using the Inversion Formula, obtain $v$.

$$\left[\frac{v}{v_0} = 1+\frac{2}{\pi}\int_0^\infty e^{-\kappa u^2 t}\frac{J_0(ur)Y_0(ua)-J_0(ua)Y_0(ur)}{J_0^2(ua)+Y_0^2(ua)}\frac{du}{u}.\right.$$

15. Find the temperature in the sphere $r = a$ at time $t > 0$, if the solid is initially at zero and the surface is kept at $v_0$.

$$\left[\frac{v}{v_0} = 1+\frac{2a}{\pi r}\sum_1^\infty \frac{(-1)^n}{n} e^{-\kappa(n^2\pi^2/a^2)t}\sin\frac{n\pi}{a}r.\right.$$

16. Find the temperature due to a unit instantaneous spherical surface source at $t = 0$ over $r = r'$ in the sphere $r = a$, the solid being then at zero and the surface thereafter kept at zero.

$$\left[v = \frac{1}{2\pi a r r'}\sum_1^\infty e^{-\kappa(n^2\pi^2/a^2)t}\sin\frac{n\pi}{a}r\sin\frac{n\pi}{a}r'.\right.$$

17. Find the temperature due to a unit instantaneous spherical surface source at $t = 0$ over $r = r'$ in the sphere $r = a$, when radiation takes place at the surface into a medium at zero.

$$\left[v = \frac{1}{2\pi arr'}\sum_{\alpha} e^{-\kappa\alpha^2 t}\frac{\alpha^2a^2+(ah-1)^2}{\alpha^2a^2+ah(ah-1)}\sin\alpha r\sin\alpha r',\right.$$

the summation being taken over the positive roots of

$$\alpha a\cos\alpha a+(ah-1)\sin\alpha a = 0.$$

18. A solid is bounded internally by the sphere $r = a$ and extends to infinity. The surface $r = a$ is kept at a constant temperature $v_0$ and the initial temperature is zero. Find the temperature $v$ at the time $t$.

$$\left[v = \frac{av_0}{r}\left\{1-\operatorname{erf}\frac{(r-a)}{2\sqrt{(\kappa t)}}\right\}.\right.$$

19. Find the temperature due to a unit instantaneous spherical surface source generated at $t = 0$ over $r = r'$ in the solid bounded internally by the sphere $r = a$ and extending to infinity, the surface $r = a$ being kept at zero for $t > 0$.

$$\left[v = \frac{1}{8\pi rr'\sqrt{(\pi\kappa t)}}\{e^{-(r-r')^2/4\kappa t}-e^{-(r+r'-2a)^2/4\kappa t}\}.\right.$$

20. Find the temperature in the solid of Ex. 19 due to a unit instantaneous spherical surface source at $t = 0$ over $r = r'$, when radiation takes place at the surface $r = a$ into a medium at zero.

$$\left[v = \frac{1}{8\pi rr'\sqrt{(\pi\kappa t)}}\times\right.$$

$$\times\left\{e^{-(r-r')^2/4\kappa t}+e^{-(r+r'-2a)^2/4\kappa t}-2\left(\frac{1}{a}+h\right)\int\limits_0^\infty e^{-(1/a+h)\xi-(r+r'-2a+\xi)^2/4\kappa t}\,d\xi\right\}.$$

21. A heavy uniform string of length $l$ and line density $\rho$ is stretched between two fixed points, $x = 0$ and $x = l$, to tension $\rho c^2$. It is plucked a small distance $b$ at a point distant $a$ from the origin and released at $t = 0$. Show that its subsequent displacement is

$$\frac{2bl^2}{\pi^2a(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}.$$

22. A uniform string of density $\rho$ and length $l$ is stretched to tension $\rho c^2$ between fixed points, $x = 0$ and $x = l$. The string is plucked a small distance $b$ at the point $x = \frac{1}{3}l$ and released at $t = 0$. Show that the subsequent form of the string is given for $0 < t \leqslant l/3c$ by

$$3bx/l \quad \text{when } x \leqslant \tfrac{1}{3}l-ct,$$

$$\frac{3b}{4l}x+\frac{9b}{4l}(\tfrac{1}{3}l-ct) \quad \text{when } \tfrac{1}{3}l-ct < x < \tfrac{1}{3}l+ct,$$

and $$\frac{3b}{2l}(l-x) \quad \text{when } l > x > ct+\tfrac{1}{2}l.$$

Determine also the form of the string in the intervals $\tfrac{1}{2}l < ct \leqslant \tfrac{3}{2}l$ and $\tfrac{3}{2}l < ct \leqslant l$.

23. A heavy uniform string of length $l$ and line density $\rho$ is stretched between two fixed points, $x = 0$ and $x = l$, to tension $\rho c^2$. At $t = 0$, when the string is straight and at rest, a blow is struck at $x = a$ so that momentum $\sigma$ is communicated to a small region about $x = a$. Show that the subsequent form of the string is

$$\frac{2\sigma}{\pi\rho c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}.$$

24. An indefinitely long string $x > 0$ of line density $\rho$ is stretched to tension $\rho c^2$ and is at rest in its equilibrium position. For $t > 0$ the end $x = 0$ is given a small oscillation $a\sin\omega t$. Show that the displacement at the point $x$ is given by

$$\begin{aligned} a\sin\omega(t-x/c) &\quad \text{if } t > x/c,\\ 0 &\quad \text{if } t < x/c. \end{aligned}$$

25. A string of density $\rho$ and length $l$ is stretched to tension $\rho c^2$. The end $x = 0$ is fixed and the end $x = l$ is attached to a massless ring free to slide on a smooth rod. At $t = 0$, when the system is at rest with the ring displaced a small distance $a$ from the equilibrium position, the ring is released. Show that the subsequent displacement of the string is

$$\frac{8a}{\pi^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\sin\frac{(2r+1)\pi x}{2l}\cos\frac{(2r+1)\pi ct}{2l}.$$

26. A string of density $\rho$ and length $l$ is stretched to tension $\rho c^2$. The end $x = l$ is fixed and at $t = 0$, when the string is at rest in its equilibrium position, the end $x = 0$ is given a small oscillation $a\sin\omega t$. Show that the subsequent displacement of the point $x$ is

$$\frac{a\sin\omega t\sin\omega(l-x)/c}{\sin\omega l/c}+\sum_{r=1}^{\infty}\frac{2lca\omega}{\omega^2l^2-\pi^2r^2c^2}\sin\frac{r\pi x}{l}\sin\frac{r\pi ct}{l}.$$

27. A uniform string of length $2l$ and density $\rho$ has a particle of mass $m$ attached to its middle point and is stretched to tension $\rho c^2$ between the fixed points $x = \pm l$. At $t = 0$, when the string is straight and at rest, the particle is set in motion by a transverse impulse $I$. Show that its subsequent displacement is

$$\frac{2lI}{mc}\sum_{n=1}^{\infty}\frac{1}{\alpha_n(1+k\operatorname{cosec}^2\alpha_n)}\sin\frac{\alpha_n ct}{l},$$

where $k = 2\rho l/m$ and $\alpha_n$, $n = 1, 2,\ldots$, are the positive roots of $k\cot\alpha = \alpha$.

28. A heavy uniform string of length $3l$ and line density $\rho$ is fixed at the ends and a particle of mass $m$ is attached at a distance $l$ from one end. The tension is $\rho c^2$. A transverse velocity $v$ is given to the particle at $t = 0$. Show that the subsequent displacement $x$ of the particle is given by

$$\bar{x} = \frac{mv\sinh(pl/c)\sinh(2pl/c)}{mp^2\sinh(pl/c)\sinh(2pl/c)+\rho cp\sinh(3pl/c)},$$

and evaluate $x$ up to the time $4l/c$.

29. A pipe, open at one end and closed at the other, is suddenly brought to rest at time $t = 0$ after having been for some time in motion with uniform velocity $v$ parallel to the length of the pipe. Show that the subsequent displacement of the air at the point distant $x$ from the closed end is

$$\frac{8vl}{c\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l}\sin\frac{(2n+1)\pi ct}{2l}.$$

30. In the problem of Ex. 29 show that the displacement is also given by

$$\begin{aligned} &vt, && 0 < ct < x,\\ &vx/c, && x < ct < 2l-x,\\ v(2l-&ct)/c, && 2l-x < ct < 2l+x,\\ -&vx/c, && 2l+x < ct < 4l-x,\\ &&& \text{etc.}\end{aligned}$$

31. A closed pipe of length $2l$ contains air whose density is greater than that outside in the ratio $1+\epsilon : 1$, where $\epsilon$ is small. At $t = 0$, when the air is at rest, the ends $x = \pm l$ of the pipe are opened. Show that the velocity potential at the point $x$ at time $t$ is

$$\frac{8\epsilon cl}{\pi^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\cos\frac{(2r+1)\pi x}{2l}\sin\frac{(2r+1)\pi ct}{2l}.$$

32. Fluid is contained in a long straight tube closed at one end $x = 0$. When $t = 0$ the fluid is everywhere at rest while the condensation $s$ is $s_0$ (constant) for values of $x$ between 0 and $a$ and zero elsewhere. Determine $s$ for all $x$, $t$; draw $(s,t)$ graphs for the values $\frac{1}{2}a$ and $\frac{3}{2}a$ of $x$ and explain the differences between them.

33. A uniform bar of length $2l$ is compressed by forces applied at its ends so that its length is $2l(1-\epsilon)$. At $t = 0$ the forces are released. Show that the subsequent displacement of the point $x$ is

$$\frac{8\epsilon l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l}\cos\frac{(2n+1)\pi ct}{2l},$$

where the origin is at the middle point of the bar and $c$ is the velocity of longitudinal waves in the bar.

34. A uniform bar $0 < x < l$ of unit area, mass $m$, and length $l$ is at rest on a smooth horizontal plane, when at $t = 0$ constant force $P$ is applied at the end $x = l$ in the direction of the length of the bar. Show that the subsequent displacement of the point of the bar originally at $x$ is

$$\frac{Pt^2}{2m}+\frac{2Pl^2}{\pi^2c^2m}\sum^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}\left(1-\cos\frac{n\pi ct}{l}\right),$$

where $c$ is the velocity of longitudinal waves in the bar.

35. A uniform bar of length $l$ is at rest and unstrained when at $t = 0$ the end $x = 0$ is given a forced oscillation $a\sin\omega t$. Show that the motion of the end $x = l$ is given by

$$\begin{aligned} &0, && 0 < ct < l,\\ &2a\sin\omega(t-l/c), && l < ct < 3l,\\ &2a[\sin\omega(t-l/c)-\sin\omega(t-3l/c)], && 3l < ct < 5l. \end{aligned}$$

36. A uniform bar of unit area, mass $m$, and length $l$, initially at rest in the position $0 < x < l$ and unstrained, is given a longitudinal blow $P$ at the end $x = 0$ at time $t = 0$. Show that the displacement of the other end is

$$\begin{aligned} &0, && 0 < ct < l,\\ &2cP/E, && l < ct < 3l,\\ &4cP/E, && 3l < ct < 5l, \end{aligned}$$

and find the displacement at any point.

37. A uniform bar is hanging vertically from a fixed point, and stretched under its own weight, when a concentrated load is suddenly attached at the lower end of the bar; obtain equations to determine the stress at the fixed end at any subsequent time.

In particular, if the weight of the load is equal to the weight of the bar, prove that the instant of maximum stress is given by

$$ct/l = 3+\tfrac{1}{2}(1+1/e^2),$$

where $l$ is the length of the bar and $c$ is the velocity of extensional waves in the bar.

38. A bar of length $l$ with the end $x = 0$ fixed is struck at the other end by a particle of $1/n$ times its mass moving with velocity $V$ in the direction of the length of the bar. Show that the pressure on the struck end of the bar is

$$(EV/c)e^{-kt}, \quad 0 < ct < 2l,$$

$$(EV/c)e^{-kt}+2(EV/c)\{1-k(t-2l/c)\}e^{-k(t-2l/c)}, \quad 2l < ct < 4l,$$

$$(EV/c)e^{-kt}+2(EV/c)\{1-k(t-2l/c)\}e^{-k(t-2l/c)}+$$
$$+2(EV/c)\{1-3k(t-4l/c)+k^2(t-4l/c)^2\}e^{-k(t-4l/c)}, \quad 4l < ct < 6l,$$

where $c$ is the velocity of longitudinal waves in the bar, and $k = nc/l$.

39. A circular membrane of radius $a$ and density $\rho$ is stretched to tension $T$. At $t = 0$ a uniform normal pressure $P$ per unit area is applied to the surface. Show that its subsequent displacement is

$$\frac{P}{T}\left\{\tfrac{1}{4}(a^2-r^2)-2a^2\sum\frac{J_0(\omega r/a)}{\omega^3 J_1(\omega)}\cos\left(\frac{\omega ct}{a}\right)\right\},$$

where $c^2 = T/\rho$, and the summation extends to all roots of $J_0(\omega) = 0$.

40. A semi-infinite transmission line $x > 0$ has resistance $R$ and capacity $C$ per unit length and zero initial current and charge. At $t = 0$ constant E.M.F. $E_0$ is applied at $x = 0$ through resistance $R_0$. Show that the potential at $x$ is

$$E_0\{1-\operatorname{erf}\tfrac{1}{2}x\sqrt{(RC/t)}\}-E_0\,e^{x\sqrt{(RCk)}+kt}\{1-\operatorname{erf}[\tfrac{1}{2}x\sqrt{(RC/t)}+\sqrt{(kt)}]\},$$

where $k = R/(CR_0^2)$.

41. In the problem of Ex. 40 show that the current at $x = 0$ is

$$\frac{E_0}{R_0}e^{kt}\{1-\operatorname{erf}\sqrt{(kt)}\}.$$

42. A semi-infinite transmission line $x > 0$ has resistance $R$ and capacity $C$ per unit length and zero initial current and charge. At $t = 0$ unit E.M.F. is applied at $x = 0$ through capacity $C_0$. Show that the current at $x$ is

$$C_0\sqrt{(\kappa/\pi t)}e^{-x^2RC/4t}-C_0\,\kappa e^{x\sqrt{(RC\kappa)}+\kappa t}\{1-\operatorname{erf}[\tfrac{1}{2}x\sqrt{(RC/t)}+\sqrt{(\kappa t)}]\},$$

where $\kappa = C/RC_0^2$.

43. A semi-infinite transmission line has resistance $R$, capacity $C$, and leakage conductance $G$ per unit length. At $t = 0$ unit E.M.F. is applied at $x = 0$ from zero initial conditions. Show that the potential at $x$ is

$$\tfrac{1}{2}e^{-y\sqrt{(2\beta)}}\left\{1-\operatorname{erf}\left[\frac{y}{2\sqrt{t}}-\sqrt{(2\beta t)}\right]\right\}+\tfrac{1}{2}e^{y\sqrt{(2\beta)}}\left\{1-\operatorname{erf}\left[\frac{y}{2\sqrt{t}}+\sqrt{(2\beta t)}\right]\right\},$$

where $y = x\sqrt{(RC)}$ and $\beta = G/2C$.

44. A cable of resistance $R$ and capacity $C$ per unit length is earthed at $x = l$. At $t = 0$, when there is no current and charge in the cable, constant E.M.F. $E$ is applied through resistance $R_1$ at $x = 0$. Show that the potential of the point $x$ is given by

$$\frac{ER(l-x)}{R+R_1}+2ER^2\sum_{n=1}^{\infty}e^{-\alpha_n^2 t/RC}\frac{\sin\alpha_n(l-x)}{\alpha_n\{R(R_1+Rl)+lR_1^2\alpha_n^2\}\cos l\alpha_n},$$

where $\alpha_n$, $n = 1, 2,\ldots$, are the positive roots of

$$R\tan l\alpha+\alpha R_1 = 0.$$

45. A cable of resistance $R$ and capacity $C$ per unit length is earthed at $x = l$. At $t = 0$, when there is no current and charge in the cable, constant E.M.F. $E$ is applied through capacity $C_1$ at $x = 0$. Show that the potential of the point $x$ is given by

$$2R^2C_1^2 E \sum_{n=1}^{\infty} \alpha_n e^{-\alpha_n^2 t/RC} \frac{\sin \alpha_n(l-x)}{(C_1^2 R^2 l\alpha_n^2 + R^2 CC_1 + R^2 C^2 l)\cos l\alpha_n},$$

where $\alpha_n$, $n = 1, 2,\ldots$, are the positive roots of $\alpha \tan \alpha l = C/C_1$.

46. A distortionless line ($R/L = G/C$) of length $l$ is earthed at $x = l$ and has zero initial current and charge. At $t = 0$ E.M.F. $\sin \omega t$ is applied at $x = 0$. Show that the potential at $x$ is

$$e^{-\rho x/v} \sin \omega(t-x/v), \quad 0 < t < x/v,$$

$$e^{-\rho x/v} \sin \omega(t-x/v) + e^{-\rho(2l-x)/v} \sin \omega\{t-(2l-x)/v\}, \quad x/v < t < (2l-x)/v,$$

etc.,

where $\rho = R/L$ and $v = (LC)^{-\frac{1}{2}}$.

47. A distortionless line ($R/L = G/C$) of length $l$ is initially charged to unit potential. The end $x = l$ is insulated and at $t = 0$ the end $x = 0$ is earthed. Show that the potential at $x$ is

$$\frac{4}{\pi} e^{-\rho t} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin \frac{(2n+1)\pi x}{2l} \cos \frac{(2n+1)\pi vt}{2l},$$

where $\rho = R/L$, $v = (LC)^{-\frac{1}{2}}$.

48. A finite transmission line $0 < x < l$ has none of $R$, $L$, $G$, $C$ zero. The initial current and potential are zero. The end $x = l$ is insulated. At $t = 0$ constant E.M.F. $E$ is applied at $x = 0$. Show that the current at $x$ is

$$E\sqrt{\left(\frac{G}{R}\right)} \frac{\sinh(l-x)\sqrt{(RG)}}{\cosh l\sqrt{(RG)}} + \\ + \frac{\pi^2 v^2 E}{2l^3} e^{-\rho t} \sum_{n=0}^{\infty} \frac{(2n+1)^2 \cos(\nu_n t - \theta_n - \phi_n)\cos(2n+1)\pi x/2l}{\nu_n(\nu_n^2+\rho^2)^{\frac{1}{2}}[(R-L\rho)^2+L^2\nu_n^2]^{\frac{1}{2}}},$$

where the notation is that of § 87 and

$$\tan \theta_n = \rho/\nu_n, \qquad \tan \phi_n = L\nu_n/(R-L\rho).$$

49. A transmission line of length $l$ has zero initial charge and current. The ends $x = 0$ and $x = l$ are kept at unit and zero potential respectively for $t > 0$. Show that the potential at the point $x$ is

$$\frac{\sinh(l-x)\sqrt{(RG)}}{\sinh l\sqrt{(RG)}} - \frac{2\pi v^2}{l^2} e^{-\rho t} \sum_{n=1}^{\infty} \frac{n \cos(\nu_n t - \theta_n)\sin n\pi x/l}{\nu_n(\nu_n^2+\rho^2)^{\frac{1}{2}}},$$

where $\nu_n = \{n^2\pi^2 v^2/l^2 - \sigma^2\}^{\frac{1}{2}}$, $\tan \theta_n = \rho/\nu_n$, and the other quantities have their usual meanings.

50. Prove that the solution of

$$\kappa\frac{\partial^2 v}{\partial x^2} = k\frac{\partial v}{\partial x}+\frac{\partial v}{\partial t}, \quad t > 0,$$

with $v = 0$ when $x = 0$, $v = V$ when $x = a$, $v = 0$ when $t = 0$, is

$$V\frac{e^{kx/\kappa}-1}{e^{ka/\kappa}-1}+$$

$$+\frac{2V}{a^2}\sum_{n=1}^{\infty}\frac{(-)^n n\pi\sin n\pi x/a}{(n\pi/a)^2+(k/2\kappa)^2}\exp\{k(x-a)/2\kappa-t[(n^2\pi^2\kappa/a^2)+(k^2/4\kappa)]\}.$$

# APPENDIX I

## *Lerch's Theorem*

The Laplace Transform $f(p)$ of a given function $F(t)$ is defined by the equation

$$f(p) = \int_0^\infty e^{-pt}F(t)\,dt, \tag{1}$$

where $p$ is a positive number (or a number whose real part is positive) and the integral on the right hand converges.

It is known† that if $G(t)$ is another function which satisfies (1),

$$F(t)-G(t) = N(t),$$

where $N(t)$ is a null-function, i.e. a function such that

$$\int_0^t N(t)\,dt = 0$$

for every $t \geqslant 0$.

A particular case of a null-function is a function which is zero except at a finite number of points.

It is clear that a continuous function cannot be a null-function unless it vanishes for $t \geqslant 0$. It would thus follow that if, for a given $f(p)$, we can find a continuous function $F(t)$ satisfying (1), this is the only continuous function which satisfies it.

And again, if for a given $f(p)$ we can find a function $F(t)$ with only ordinary discontinuities satisfying (1), any function which satisfies (1) and has ordinary discontinuities at the same points as $F(t)$ can differ from $F(t)$ only at those points.

We shall prove this uniqueness theorem for the case of the continuous function by elementary methods. The proof requires the following lemma.‡

Lemma. *Let $\psi(x)$ be continuous in $0 \leqslant x \leqslant 1$ and let*

$$\int_0^1 x^{n-1}\psi(x)\,dx = 0, \quad \textit{for } n = 1, 2,\ldots.$$

*Then* $$\psi(x) \equiv 0 \quad \textit{in} \quad 0 \leqslant x \leqslant 1.$$

If $\psi(x)$ is not identically zero in the closed interval, there must be an interval (e.g. $a \leqslant x \leqslant b$, where $0 < a < b < 1$) in which $\psi(x)$ is always positive (or always negative). Take the first alternative.

From the parabola $y = (b-x)(x-a)$ we see that, if $c$ is the larger of the two numbers $ab$, $(1-a)(1-b)$, then

$$1+\frac{1}{c}(b-x)(x-a) > 1, \quad \text{when } a < x < b$$

† Doetsch, loc. cit., chap. iii, §7. The proof is rather difficult. See also Lerch, *Acta Mathematica*, **27** (1903), 339–52.

‡ Cf. Bremekamp, *Kon. Akad. v. Wet. Amsterdam*, Proceedings of the Section of Sciences, **40** (1937), 689.

and

$$0 < 1+\frac{1}{c}(b-x)(x-a) < 1, \quad \text{when } 0 < x < a \text{ and } b < x < 1.$$

Thus $[1+(1/c)(b-x)(x-a)]^r$ can be made as large as we please in $a < x < b$ by making the positive integer $r$ greater and greater, and in $0 < x < a$, $b < x < 1$ it can in the same way be made as small as we please.

But $[1+(1/c)(b-x)(x-a)]^r$ is a polynomial in $x$, and by our hypothesis $\int_0^1 x^{n-1}\psi(x)\,dx = 0$ for $n = 1, 2,\ldots$.

Therefore we should have

$$\int_0^1 \left[1+\frac{1}{c}(b-x)(x-a)\right]^r \psi(x)\,dx = 0$$

for every positive integer $r$. Whereas from the above, by choosing $r$ large enough,

$$\int_0^1 \left[1+\frac{1}{c}(b-x)(x-a)\right]^r \psi(x)\,dx > 0.$$

Thus with the first alternative we are brought to a contradiction; and a similar argument applies to the other.

It follows that $\psi(x) \equiv 0$ in $0 \leqslant x \leqslant 1$.

THEOREM. *If $f(p) = \int_0^\infty e^{-pt}F(t)\,dt$, $p > p_0$, is satisfied by a continuous function $F(t)$, there is no other continuous function which satisfies the equation.*

For, if possible, let there be another continuous function $G(t)$ satisfying the equation. Let $g(t) = F(t)-G(t)$. Then

$$\int_0^\infty e^{-pt}g(t)\,dt = 0, \quad p > p_0, \tag{2}$$

and $g(t)$ is continuous.

Let $p = p_0+n$, where $n$ is any positive integer.

Then since, by integration by parts,

$$n\int_0^\infty e^{-nt}\,dt \int_0^t e^{-p_0 t}g(t)\,dt = \int_0^\infty e^{-(p_0+n)t}g(t)\,dt,$$

it follows from (2) that

$$\int_0^\infty e^{-nt}\phi(t)\,dt = 0,$$

where $$\phi(t) = \int_0^t e^{-p_0 t}g(t)\,dt.$$

Put in (3) $x = e^{-t}$, $\psi(x) = \phi[\log(1/x)]$. Then $\psi(x)$ is continuous in the closed interval $0 \leqslant x \leqslant 1$, since we take

$$\psi(0) = \lim_{t\to\infty}\phi(t) \quad \text{and} \quad \psi(1) = \phi(0) = 0.$$

Also $$\int_0^1 x^{n-1}\psi(x)\,dx = 0, \quad n = 1, 2,\ldots.$$

It follows from the lemma that $\psi(x) = 0$ in $0 \leqslant x \leqslant 1$, and therefore

$$\phi(t) = \int_0^t e^{-p_0 t}g(t)\,dt = 0, \quad \text{when } t \geqslant 0. \tag{4}$$

Now $e^{-p_0 t}g(t)$ is continuous in $t \geqslant 0$, so it follows from (4) that $e^{-p_0 t}g(t) = 0$, when $t \geqslant 0$, i.e. $g(t) = 0$, when $t \geqslant 0$, and the theorem is proved.

It is easily seen that it is satisfied by $J_n(iz)$ and $Y_n(iz)$, but it is more convenient to take as standard solutions the following functions which can be obtained from the above by multiplication by constants.

For the solution of the first kind, we take

$$I_n(z) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{n+2r}}{r!\,\Gamma(n+r+1)}, \tag{11}$$

available for all values of $n$.

As in Bessel's equation, $I_n(z)$ and $I_{-n}(z)$ are independent solutions of (10), when $n$ is not an integer, but $I_n(z) = I_{-n}(z)$, when $n$ is an integer.

It is usual to take as the standard solution of the second kind, available for all values of $n$,

$$K_n(z) = \tfrac{1}{2}\pi \frac{I_{-n}(z) - I_n(z)}{\sin n\pi}, \tag{12}$$

the limit being taken, as in § 2, when $n$ is an integer.

With this definition of $K_n(z)$, we have

$$K_0(z) = -I_0(z)\{\log(\tfrac{1}{2}z)+\gamma\}+(\tfrac{1}{2}z)^2+(1+\tfrac{1}{2})\frac{(\frac{1}{2}z)^4}{(2!)^2}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(\frac{1}{2}z)^6}{(3!)^2}+\dots \tag{13}$$

and, when $n$ is any positive integer,

$$K_n(z) = (-1)^{n+1}I_n(z)\{\log(\tfrac{1}{2}z)+\gamma\}+$$
$$+\tfrac{1}{2}(-1)^n \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!}\Big[\sum_{m=1}^{n+r} m^{-1}+\sum_{m=1}^{r} m^{-1}\Big]$$
$$+\tfrac{1}{2}\sum_{r=0}^{n-1}(-1)^r(\tfrac{1}{2}z)^{-n+2r}\frac{(n-r-1)!}{r!}, \tag{14}$$

where for $r = 0$, we write $\sum_{m=1}^{n} m^{-1}$ in place of $\Big(\sum_{m=1}^{n+r} m^{-1}+\sum_{m=1}^{r} m^{-1}\Big)$.

Since $K_1(z) = -K_0'(z)$, it is easy to write down $K_1(z)$ from the expression in (14) for $K_0(z)$.

These solutions have the important properties†

$$I_n(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}}\Big\{1+O\Big(\frac{1}{z}\Big)\Big\}+\frac{e^{-z+(n+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}}\Big\{1+O\Big(\frac{1}{z}\Big)\Big\}, \tag{15}$$

when $-\tfrac{1}{2}\pi < \arg z < \tfrac{3}{2}\pi$,

$$\frac{e^z}{(2\pi z)^{\frac{1}{2}}}\Big\{1+O\Big(\frac{1}{z}\Big)\Big\}+\frac{e^{-z-(n+\frac{1}{2})\pi i}}{(2\pi z)^{\frac{1}{2}}}\Big\{1+O\Big(\frac{1}{z}\Big)\Big\}, \tag{16}$$

when $-\tfrac{3}{2}\pi < \arg z < \tfrac{1}{2}\pi$.

$$K_n(z) = \sqrt{\Big(\frac{\pi}{2z}\Big)}e^{-z}\Big\{1+O\Big(\frac{1}{z}\Big)\Big\}, \quad \text{when } |\arg z| < \tfrac{3}{2}\pi, \tag{17}$$

so that when $\mathbf{R}(z) \to \infty$, $\sqrt{z}K_n(z)$ tends exponentially to zero.

† *W.B.F.*, § 7.23.

6. *Some important properties of the Bessel functions which have been used in the text are collected here for reference.*

*Recurrence relations:*

$$zI'_n(z)+nI_n(z) = zI_{n-1}(z), \tag{18}$$
$$zI'_n(z)-nI_n(z) = zI_{n+1}(z), \tag{19}$$
$$zK'_n(z)+nK_n(z) = -zK_{n-1}(z), \tag{20}$$
$$zK'_n(z)-nK_n(z) = -zK_{n+1}(z), \tag{21}$$
$$zJ'_n(z)+nJ_n(z) = zJ_{n-1}(z), \tag{22}$$
$$zJ'_n(z)-nJ_n(z) = -zJ_{n+1}(z). \tag{23}$$

$Y_n(z)$ satisfies the same relations as $J_n(z)$.

*Wronskian relations:*

$$J_n(z)Y'_n(z)-Y_n(z)J'_n(z) = \frac{2}{\pi z}, \tag{24}$$

$$I_n(z)K'_n(z)-K_n(z)I'_n(z) = -\frac{1}{z}. \tag{25}$$

*Functions of argument $ze^{m\pi i}$:*

$$J_n(ze^{m\pi i}) = e^{mn\pi i}J_n(z), \tag{26}$$
$$Y_n(ze^{m\pi i}) = e^{-mn\pi i}Y_n(z)+2i\sin mn\pi\cot n\pi J_n(z), \tag{27}$$
$$K_n(z) = \tfrac{1}{2}\pi i e^{\frac{1}{2}n\pi i}H_n^{(1)}(iz), \tag{28}$$
$$\left.\begin{aligned} I_n(z) &= e^{-\frac{1}{2}n\pi i}J_n(ze^{\frac{1}{2}\pi i}), \quad -\pi < \arg z \leqslant \tfrac{1}{2}\pi, \\ &= e^{\frac{3}{2}n\pi i}J_n(ze^{-\frac{3}{2}\pi i}), \quad \tfrac{1}{2}\pi < \arg z \leqslant \pi, \end{aligned}\right\} \tag{29}$$
$$I_0(ze^{\pm\frac{1}{2}i\pi}) = J_0(z), \tag{30}$$
$$K_0(ze^{\pm\frac{1}{2}i\pi}) = \mp\tfrac{1}{2}\pi i\{J_0(z)\mp iY_0(z)\}, \tag{31}$$
$$I_1(ze^{\pm\frac{1}{2}i\pi}) = \pm iJ_1(z), \tag{32}$$
$$K_1(ze^{\pm\frac{1}{2}i\pi}) = -\tfrac{1}{2}\pi[J_1(z)\mp iY_1(z)]. \tag{33}$$

7. Finally we derive formally two results required in the text. For a complete proof along these lines justifying the steps made here see *W.B.F.*, § 6.2, 6.22, or *G. and M.*, p. 53 (43).

Since
$$\frac{1}{p^{m+1}} = \frac{1}{m!}\int_0^\infty e^{-pt}t^m\,dt, \quad m \geqslant 0,$$

it follows from the Inversion Theorem that

$$\frac{t^m}{m!} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{d\lambda}{\lambda^{m+1}}, \quad m \geqslant 0.$$

Now
$$I_n(z) \quad \sum_{r=0}^{\infty}\frac{(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!}.$$

Introducing in this the result above with $l=1$ and $n+r$ written for $m$, and assuming that we may invert the orders of integration and summation, we have

$$I_n(z) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{n+2r}}{2\pi i r!} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda}\,d\lambda}{\lambda^{n+r+1}}$$

$$= \frac{1}{2\pi i}(\tfrac{1}{2}z)^n \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda}\,d\lambda}{\lambda^{n+1}} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{2r}}{\lambda^r r!}$$

$$= \frac{(\frac{1}{2}z)^n}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda+z^2/4\lambda} \frac{d\lambda}{\lambda^{n+1}}. \tag{34}$$

In this result put $$\frac{2\lambda}{z} = u+\sqrt{(u^2-1)},$$

so that $$\frac{z}{2\lambda} = u-\sqrt{(u^2-1)}.$$

The contour becomes one which may be deformed into $(\gamma-i\infty, \gamma+i\infty)$ and we obtain

$$I_n(z) = \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{zu} \frac{[u-\sqrt{(u^2-1)}]^n\,du}{\sqrt{(u^2-1)}}.$$

It follows that

$$I_n(az) = \frac{1}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{z\lambda} \frac{[\lambda-\sqrt{(\lambda^2-a^2)}]^n\,d\lambda}{a^n\sqrt{(\lambda^2-a^2)}},$$

and hence by the Inversion Theorem†

$$\int\limits_0^{\infty} e^{-pz} I_n(az)\,dz = \frac{[p-\sqrt{(p^2-a^2)}]^n}{a^n\sqrt{(p^2-a^2)}}. \tag{35}$$

† Cf. *W.B.F.*, §13.2 (8).

# APPENDIX III

## *Impulsive Functions*

THE so-called impulsive functions have not been used in the body of this book, since a careful treatment of them presents considerable difficulties. It is desirable that the student should have some knowledge of them, both because they provide a convenient idealization of certain types of problem of practical importance, and because they give a physical interpretation to the solution in cases in which the previous methods are not applicable. We take these applications in order.

The Dirac $\delta$ function is defined† to be zero if $x \neq 0$ and to be infinite at $x = 0$ in such a way that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{1}$$

It is here regarded as a concise way of expressing the idea of a function which is very large in a very small region, zero outside this region, and has unit area. Several continuous functions have been used which possess this property in the limit as a parameter tends to zero, but we shall here take the simple function

$$\begin{aligned} \delta(x) &= 0, && x \leqslant 0, \\ &= 1/\epsilon, && 0 < x < \epsilon, \\ &= 0, && x \geqslant \epsilon, \end{aligned} \tag{2}$$

where $\epsilon$ may be made as small as we please. This function clearly possesses the property (1). The only other result we require is the following:

*If $f(x)$ is a continuous function,*

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a). \tag{3}$$

This follows since, from (2),

$$\int f(x)\delta(x-a)\, dx = \frac{1}{\epsilon}\int_{a}^{a+\epsilon} f(x)\, dx$$

$$= f(a+\theta\epsilon), \quad 0 < \theta < 1,$$

and, since $f(x)$ is continuous, the right-hand side $\rightarrow f(a)$ as $\epsilon \rightarrow 0$.

† Dirac, *Quantum Mechanics*, 1st ed., p. 64. $\delta(x)$ is there taken to be an even function, but a definition of type (2) is more convenient for the present applications.

A particular case of (3) is

$$\int_0^\infty e^{-px}\,\delta(x)\,dx = 1, \tag{4}$$

i.e. the Laplace Transform of the $\delta$ function is unity.

*Mechanical applications.*

Unit instantaneous impulse at $t = 0$ in the mechanical sense may now be regarded as due to force $\delta(t)$ and treated in the usual way, as in the following examples.

Ex. 1. *Instantaneous impulse $P$ is applied at $t = 0$ to a particle of mass $m$ at rest at the origin.*

The equation of motion is

$$mD^2x = P\,\delta(t).$$

Thus the subsidiary equation is

$$mp^2\bar{x} = P.$$

Therefore $$x = Pt/m.$$

Ex. 2. *A uniform rod of mass $m$ and length $2a$ is at rest on a smooth horizontal plane when at $t = 0$ instantaneous impulse $P$ is given at one end in a direction perpendicular to the rod.*

Let $\theta$ be the angular displacement of the rod and $x$ the linear displacement of its middle point. The equations of motion are

$$mD^2x = P\,\delta(t),$$
$$\tfrac{1}{3}ma^2D^2\theta = Pa\,\delta(t).$$

The subsidiary equations are $mp^2\bar{x} = P$ and $\frac{1}{3}map^2\bar{\theta} = P$.

Therefore $$x = Pt/m, \qquad \theta = 3Pt/ma.$$

Ex. 3. *A particle of mass $m$ oscillates in a straight line under restoring force $mn^2$ times the displacement. The particle is at rest in its equilibrium position at $t = 0$. Impulses $P$ are given at $t = rT$, $r = 0, 1, 2, \ldots$. Find the motion of the particle.*

The equation of motion is

$$mD^2x + mn^2x = P\sum_{r=0}^{\infty}\delta(t-rT).$$

Thus, using Chapter I, § 3, Theorem V, the subsidiary equation is

$$m(p^2+n^2)\bar{x} = P\sum_{r=0}^{\infty}e^{-rpT} = \frac{P}{1-e^{-pT}}.$$

Hence $$\bar{x} = \frac{P}{m(p^2+n^2)(1-e^{-pT})}.$$

Therefore, using the Inversion Theorem,

$$\frac{P}{2\pi im}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}\,d\lambda}{(\lambda^2+n^2)[1-e^{-\lambda T}]}.$$

The integrand has simple poles when $\lambda = \pm in$, and when $\lambda = \pm 2r\pi i/T$, $r = 0, 1, 2,\ldots$, provided $T$ is not an integral multiple of $2\pi/n$. Evaluating the residues at these poles we obtain, in the usual way,

$$x = -\frac{P\cos n(t+\frac{1}{2}T)}{2mn\sin\frac{1}{2}nT}+\frac{P}{mn^2T}+\frac{2PT}{m}\sum_{r=1}^{\infty}\frac{1}{(n^2T^2-4r^2\pi^2)}\cos\frac{2r\pi t}{T}.$$

*Electrical applications.*

In electric-circuit theory an impulsive E.M.F. $E_0\,\delta(t)$, typifying a very large voltage applied for a very short time, so that the time-integral of the E.M.F. is $E_0$, is of interest.

Ex. 4. *Such an E.M.F. applied to the circuit of* § 13 *with zero initial conditions* gives the subsidiary equation

$$\Big(Lp+R+\frac{1}{Cp}\Big)I = E_0.$$

Thus
$$I = \frac{E_0\,p}{L[(p+\mu)^2+n^2]},$$

in the notation of § 14 (1). Therefore, in the case $n^2 > 0$,

$$I = \frac{E_0}{nL}e^{-\mu t}(n\cos nt-\mu\sin nt).$$

Ex. 5. If the problem is that of *the circuit of* § 13 *with zero initial conditions excited by impulsive E.M.F.s $E_0$ applied at $t = rT$, $r = 0, 1, 2,\ldots$,* the subsidiary equation is

$$\Big(Lp+R+\frac{1}{Cp}\Big)I = E_0\sum_{r=0}^{\infty}e^{-rpT} = \frac{E_0}{1-e^{-pT}},$$

and $I$ is obtained on proceeding as in Ex. 3 above.

*Application to a space variable.*

The $\delta$ function may also be used with space variables when concentrated loads or disturbances have to be considered. For example, the forces $P(x,t)$ considered in the examples of § 65 are all of this type, being $W\delta(x-x')$, $P_0\sin\omega t\,\delta(x-x')$, $W\delta(x-vt)$ in Examples 2, 3, 4 of that section respectively. Parts of this and other sections could have been written more shortly by using the present notation and (3) above.

When a formula has been obtained involving an initial distribution $f(x)$ of some quantity, the result for a concentrated distribution at $x'$ is obtained by putting $\delta(x-x')$ for $f(x)$ and using (3). For example, from § 92 (12) it follows that the potential at any point of a doubly infinite distortionless transmission line due to initial potential $\delta(x-x')$ and zero initial current is

$$\tfrac{1}{2}e^{-\rho t}[\delta(x-x'+vt)+\delta(x-x'-vt)].$$

*The function whose Laplace Transform is a constant.*

All the applications above may be regarded as justifiable along the lines by which (3) was deduced. There is another type of application which we do not attempt to justify, but which may nevertheless be useful as providing an indication of the solution in cases to which the ordinary methods are not applicable. The most important case of this type is that in which the Laplace Transform of the solution of the problem turns out to be a constant. We have seen above that the Laplace Transform of $\delta(t)$ is 1, the inverse statement is that the function whose transform is unity is a constant.† Consider the following examples.

Ex. 6. *Constant E.M.F. $E_0$ applied to a pure capacity $C$.*

The subsidiary equation, § 13 (5), is

$$\frac{1}{Cp}\bar{I} = \frac{E_0}{p}.$$

Thus $\bar{I} = CE_0$ and $I = CE_0\,\delta(t)$, that is, the charging current is instantaneously infinite.

Ex. 7. *The filter circuit of* § 19 *with $z' = 1/Cp$, $z = R$, $z'' = 0$, $V = E$ (constant).*

Here § 19 (10) gives

$$\bar{I}_r = \frac{E\cosh(m-r)\theta}{Rp\sinh\theta\sinh m\theta},$$

where $\cosh\theta = 1+1/(2RCp)$. On evaluating the numerator and denominator of $\bar{I}_r$, as in § 19, Ex. 1, it will be found that they are of the same degree in $p$, and in fact

$$\bar{I}_r = \frac{CE}{m} + \frac{mE\cosh(m-r)\theta - CERp\sinh\theta\sinh m\theta}{mRp\sinh\theta\sinh m\theta}.$$

Then, proceeding as in § 19, we obtain

$$I_r = \frac{CE}{m}\delta(t) + \frac{E(-1)^r}{4mR}e^{-\beta_m t} + \frac{E}{mR}\sum_{s=1}^{m-1}\frac{\cos rs\pi/m}{1-\cos s\pi/m}e^{-\beta_s t},$$

where $\beta_m = 1/[2RC(1-\cos s\pi/m)]$.

Ex. 8. *In the problem of* § 16, *Ex.* 4, *it is required to find the potential difference, $V$, over $CD$.*

† It will be remembered that we have only found the function whose Laplace Transform is $p^n$, where $n < 0$. The case $n = 0$ is now recognized as $\delta(t)$ and in a fuller treatment $n = 1, 2, 3,\ldots$ would appear as the successive derivatives of $\delta(t)$. These, however, rarely appear in practice.

This is given by

$$\bar{V} = (L'p+R')\bar{I} = \frac{E(L'p+R')(Lp+R)}{Rp[(L+L')p+(R+R')]}$$

$$= \frac{ELL'}{R(L+L')} + \frac{ER'}{(R+R')p} +$$

$$+ \frac{E(RL'-LR')^2}{R(L+L')(R+R')[(L+L')p+R+R']}.$$

Therefore

$$V = \frac{ELL'}{R(L+L')}\,\delta(t) + \frac{ER'}{R+R'} + \frac{E(RL'-LR')^2}{R(L+L')^2(R+R')}\,e^{-(R+R')t/(L+L')}.$$

# APPENDIX IV

## *Two-point Boundary Value Problems for Ordinary Linear Differential Equations*

THROUGHOUT this book we have been concerned with the solution of the differential equation $\phi(D)y = F(x)$, $x > 0$, with $y$, $Dy$,... given when $x = 0$. This is the typical problem of dynamics: given the forces acting on a system and the initial conditions, to find the subsequent motion. In other fields of applied mathematics other types of problem arise: (i) the range of the independent variable may be finite, $0 < x < a$, with $y$, $Dy$,... all given at $x = 0$; (ii) the range of the independent variable may be finite, $0 < x < a$, with some of $y$, $Dy$,... given at $x = 0$ and some at $x = a$; (iii) the range of the independent variable may be finite and not the actual values of $y$, $Dy$,..., but linear connexions of them may be specified, some at $x = 0$ and some at $x = a$.

In the classical method of solving differential equations the distinction between these types is not so apparent. The general solution is obtained as a sum of the particular integral and the complementary function, the latter containing $n$ arbitrary constants. The boundary conditions, whether at $x = 0$ or $x = a$, furnish sufficient equations to determine the arbitrary constants.

The question naturally arises whether the methods of Chapter I can be applied to problems of the above types and whether it is worth while doing so.

Firstly it must be remarked that the Laplace Transformation method gives a solution in the same form as the classical method, i.e. as a sum of a particular integral,† the function whose transform is $F(p)/\phi(p)$, and a number of terms involving the initial conditions.

Now suppose we have to solve the differential equation $\phi(D)y = F(x)$ in $0 < x < a$ with some of $y$, $Dy$,... given at $x = 0$ and some at $x = a$. We consider the differential equation $\phi(D)y = F(x)$ in‡ $x > 0$ with initial values $y_0$, $y_1$,... of $y$, $Dy$,...; some of these are given and the rest are to be regarded as arbitrary constants. The conditions at $x = a$ will be sufficient to determine these.

Whether this procedure is preferable to the classical method depends largely on the problem; it has the advantage that the theorems of Chapter I may be used; this is valuable in problems in which $F(x)$ is a step or broken function. In many problems a convenient compromise is to write down the complementary function in the usual way but to find a particular integral by the methods of the Laplace Transformation,

† The particular integrals derived by the two methods may differ by terms of the complementary function.

‡ It must be assumed that the Laplace Transform of $F(x)$ exists. This restriction is, of course, not necessary in the classical solution.

which are often simpler.† An example of this procedure is given in Ex. 3 below.

Ex. 1. *Part of a uniform heavy chain of weight $w$ per unit length rests on the upper surface of a rough vertical circle of radius $a$, and part hangs vertically. If one end is at the highest point of the circle, find the greatest length that can hang freely.*

When the chain is in limiting equilibrium the tension $T$ at a point distant $a\theta$ from the highest point is given by

$$\frac{dT}{d\theta}-\mu T = wa(\mu\cos\theta-\sin\theta), \quad 0<\theta<\tfrac{1}{2}\pi,$$

where $\mu$ is the coefficient of friction.

This has to be solved with $T=0$, when $\theta=0$. Thus the subsidiary equation is

$$(p-\mu)\bar{T} = wa\frac{(\mu p-1)}{p^2+1}.$$

Therefore
$$\frac{\bar{T}}{wa} = \frac{\mu p-1}{(p-\mu)(p^2+1)}$$
$$= \frac{1}{\mu^2+1}\left\{\frac{\mu^2-1}{p-\mu}-\frac{(\mu^2-1)p-2\mu}{p^2+1}\right\}.$$

Hence
$$T = \frac{wa}{\mu^2+1}\{(\mu^2-1)e^{\mu\theta}-(\mu^2-1)\cos\theta+2\mu\sin\theta\}.$$

The greatest length which can hang freely is the tension when $\theta=\frac{1}{2}\pi$ divided by $w$, i.e.
$$\frac{a}{\mu^2+1}\{(\mu^2-1)e^{\frac{1}{2}\mu\pi}+2\mu\}.$$

Ex. 2. *A uniform beam of length $l$ has load $wx$ per unit length in $0<x<\frac{1}{2}l$ and $w(l-x)$ in $\frac{1}{2}l<x<l$. The beam is built in at $x=0$ and freely hinged at $x=l$. To find the deflexion.*

We have to solve $D^4y=f(x)$, with $f(x)=f_1(x)+f_2(x)$, say, where

$$f_1(x) = \frac{wx}{EI}, \quad x>0,$$

and
$$f_2(x) = 0, \quad 0<x<\tfrac{1}{2}l,$$
$$-\frac{2w}{EI}(x-\tfrac{1}{2}l), \quad x>\tfrac{1}{2}l.$$

Then, by § 3, Theorem V,

$$\int\limits_0^\infty e^{-px}f(x)\,dx = \frac{w}{EIp^2}-\frac{2w}{EIp^2}e^{-\frac{1}{2}pl}.$$

The boundary conditions are

$$y=Dy=0 \text{ when } x=0, \quad \text{and} \quad y=D^2y=0 \text{ when } x=l.$$

So we solve the differential equation in $x>0$ with $y$, $Dy$, $D^2y$, $D^3y$

† Cf. Jeffreys, loc. cit., § 1.8.

equal to 0, 0, $y_2$, $y_3$, when $x = 0$, where $y_2$ and $y_3$ are subsequently to be determined to make $y$ satisfy $y = D^2y = 0$ when $x = l$.

The subsidiary equation is

$$\bar{y} = \frac{y_3}{p^4}+\frac{y_2}{p^3}+\frac{w}{EIp^6}-\frac{2w}{EIp^6}e^{-\frac{1}{2}pl}.$$

Therefore, using § 3, Theorem V,

$$y = \tfrac{1}{6}x^3y_3+\tfrac{1}{2}x^2y_2+\frac{w}{EI}\frac{x^5}{5!}-\frac{2w}{EI}\begin{cases}0, & 0<x<\frac{1}{2}l,\\ \frac{1}{5!}(x-\frac{1}{2}l)^5, & x>\frac{1}{2}l.\end{cases}$$

The conditions $y = D^2y = 0$ when $x = l$ now require

$$\tfrac{1}{6}l^3y_3+\tfrac{1}{2}l^2y_2+\frac{15}{16}\frac{wl^5}{5!\,EI} = 0,$$

$$ly_3+y_2+\frac{3}{4}\frac{wl^3}{3!\,EI} = 0.$$

Thus $$y_3 = -\frac{21}{128}\frac{wl^2}{EI} \quad\text{and}\quad y_2 = \frac{5}{128}\frac{wl^3}{EI}.$$

Therefore, finally,

$$EIy = -\tfrac{7}{256}wx^3l^2+\tfrac{5}{256}wx^2l^3+\tfrac{1}{120}wx^5, \quad 0<x<\tfrac{1}{2}l,$$

$$= -\tfrac{7}{256}wx^3l^2+\tfrac{5}{256}wx^2l^3-\frac{w}{60}(x-\tfrac{1}{2}l)^5+\tfrac{1}{120}wx^5, \quad \tfrac{1}{2}l<x<l.$$

Ex. 3. *To solve*

$$\frac{d^2y}{dx^2}-a^2y = f(x), \quad 0<x<l, \tag{1}$$

*with* $y = 0$ *when* $x = 0$ *and* $x = l$.

Problems of this type have frequently arisen in Chapters VI–X as subsidiary equations of partial differential equations. They may be solved either by variation of parameters or by the use of the Green's function, or by the method sketched above.

A particular integral of (1) is the function whose Laplace Transform is

$$\frac{1}{p^2-a^2}\bar{f}(p),$$

i.e., by Theorem VI,

$$\frac{1}{a}\int_0^x f(x')\sinh a(x-x')\,dx'.$$

Adding the complementary function, $A\sinh ax+B\cosh ax$, we obtain for the complete solution of (1)

$$y = A\sinh ax+B\cosh ax+\frac{1}{a}\int_0^x f(x')\sinh a(x-x')\,dx'.$$

The conditions $y = 0$ when $x = 0$ and $x = l$ give

$$\left.\begin{aligned} B &= 0, \\ A\sinh al+\frac{1}{a}\int\limits_0^l f(x')\sinh a(l-x')\,dx' &= 0. \end{aligned}\right\}$$

Introducing these values we obtain

$$y = -\frac{1}{a\sinh al}\left\{\sinh a(l-x)\int\limits_0^x f(x')\sinh ax'\,dx'+\right.$$

$$\left.+\sinh ax\int\limits_x^l f(x')\sinh a(l-x')\,dx'\right\}.$$

# APPENDIX V

## *Table of Laplace Transforms*

$$\bar{x}(p) = \int_0^\infty e^{-pt}x(t)\,dt.$$

| | $\bar{x}(p)$ | $x(t)$ |
|---|---|---|
| 1. | $\dfrac{1}{p^{n+1}}$ | $\dfrac{t^n}{\Gamma(n+1)}$, $n > -1$ |
| 2. | $\dfrac{p}{p^2+a^2}$ | $\cos at$ |
| 3. | $\dfrac{a}{p^2+a^2}$ | $\sin at$ |
| 4. | $\dfrac{p}{p^2-a^2}$ | $\cosh at$ |
| 5. | $\dfrac{a}{p^2-a^2}$ | $\sinh at$ |
| 6. | $\dfrac{p}{(p^2+a^2)^2}$ | $\dfrac{t}{2a}\sin at$ |
| 7. | $\dfrac{a^2}{(p^2+a^2)^2}$ | $\dfrac{1}{2a}(\sin at - at\cos at)$ |
| 8. | $\dfrac{1}{p^3+a^3}$ | $\dfrac{1}{3a^2}\{e^{-at}+e^{\frac{1}{2}at}(\cos\tfrac{1}{2}\sqrt{3}at-\sqrt{3}\sin\tfrac{1}{2}\sqrt{3}at)\}$ |
| 9. | $\dfrac{p}{p^3+a^3}$ | $\dfrac{1}{3a}\{-e^{-at}+e^{\frac{1}{2}at}(\cos\tfrac{1}{2}\sqrt{3}at+\sqrt{3}\sin\tfrac{1}{2}\sqrt{3}at)\}$ |
| 10. | $\dfrac{p^2}{p^3+a^3}$ | $\tfrac{1}{3}(e^{-at}-2e^{\frac{1}{2}at}\cos\tfrac{1}{2}\sqrt{3}at)$ |
| 11. | $\dfrac{1}{p^4+4a^4}$ | $\dfrac{1}{4a^3}(\sin at\cosh at-\cos at\sinh at)$ |
| 12. | $\dfrac{p}{p^4+4a^4}$ | $\dfrac{1}{2a^2}\sin at\sinh at$ |
| 13. | $\dfrac{p^2}{p^4+4a^4}$ | $\dfrac{1}{2a}(\sin at\cosh at+\cos at\sinh at)$ |
| 14. | $\dfrac{p^3}{p^4+4a^4}$ | $\cos at\cosh at$ |
| 15. | $\dfrac{1}{p^4-a^4}$ | $\dfrac{1}{2a^3}(\sinh at-\sin at)$ |
| 16. | $\dfrac{p}{p^4-a^4}$ | $\dfrac{1}{2a^2}(\cosh at-\cos at)$ |

| | $\bar{x}(p)$ | $x(t)$ |
|---|---|---|
| 17. | $\dfrac{p^2}{p^4-a^4}$ | $\dfrac{1}{2a}(\sinh at+\sin at)$ |
| 18. | $\dfrac{p^3}{p^4-a^4}$ | $\frac{1}{2}(\cosh at+\cos at)$ |
| 19.† | $e^{-a\sqrt{p}}$ | $\dfrac{a}{2\sqrt{(\pi t^3)}}\,e^{-a^2/4t}$ |
| 20.† | $\dfrac{e^{-a\sqrt{p}}}{\sqrt{p}}$ | $\dfrac{1}{\sqrt{(\pi t)}}\,e^{-a^2/4t}$ |
| 21.† | $\dfrac{e^{-a\sqrt{p}}}{p}$ | $\dfrac{2}{\sqrt{\pi}}\int\limits_{a/2\sqrt{t}}^{\infty} e^{-u^2}\,du = 1-\operatorname{erf}\dfrac{a}{2\sqrt{t}}$ |
| 22.† | $\dfrac{e^{-a\sqrt{p}}}{p+b\sqrt{p}}$ | $e^{b^2t+ab}\left\{1-\operatorname{erf}\left(b\sqrt{t}+\dfrac{a}{2\sqrt{t}}\right)\right\}$ |
| 23.† | $K_0(a\sqrt{p})$ | $\dfrac{1}{2t}\,e^{-a^2/4t}$ |
| 24.† | $K_0(ap)$ | $\begin{cases} 0, & 0<t<a \\ (t^2-a^2)^{-\frac{1}{2}}, & t>a \end{cases}$ |
| 25.† | $\pi e^{-ap}I_0(ap)$ | $\begin{cases} 0, & t>2a \\ \{t(2a-t)\}^{-\frac{1}{2}}, & 0<t<2a \end{cases}$ |
| 26.† | $\begin{cases} I_0(a'\sqrt{p})K_0(a\sqrt{p}), & a>a' \\ I_0(a\sqrt{p})K_0(a'\sqrt{p}), & a<a' \end{cases}$ | $\dfrac{1}{2t}\,e^{-(a^2+a'^2)/4t}I_0\left(\dfrac{aa'}{2t}\right)$ $= \int\limits_0^{\infty} e^{-\alpha^2 t}\alpha J_0(\alpha a)J_0(\alpha a')\,d\alpha$ |
| 27. | $\dfrac{1}{\sqrt{(p^2+1)}}$ | $J_0(t)$ |
| 28. | $\dfrac{\{\sqrt{(p^2+1)}-p\}^n}{\sqrt{(p^2+1)}}$ | $J_n(t)$ |
| 29. | $\dfrac{1}{\sqrt{(p^2-1)}}$ | $I_0(t)$ |
| 30. | $\dfrac{\{p-\sqrt{(p^2-1)}\}^n}{\sqrt{(p^2-1)}}$ | $I_n(t)$ |
| 31.† | $\dfrac{e^{-a\sqrt{(p^2+b^2)}}}{\sqrt{(p^2+b^2)}}$ | $\begin{cases} 0, & \text{when } 0<t<a \\ J_0\{b\sqrt{(t^2-a^2)}\}, & \text{when } t>a \end{cases}$ |
| 32.† | $e^{-ap}-e^{-a\sqrt{(p^2+b^2)}}$ | $\begin{cases} 0, & \text{when } 0<t<a \\ \dfrac{abJ_1\{b\sqrt{(t^2-a^2)}\}}{\sqrt{(t^2-a^2)}}, & \text{when } t>a \end{cases}$ |

† In these $a$ is to be taken positive.

| | $\bar{x}(p)$ | $x(t)$ |
|---|---|---|
| 33.† | $\dfrac{e^{-a\sqrt{(p^2-b^2)}}}{\sqrt{(p^2-b^2)}}$ | $\begin{cases} 0, & \text{when } 0 < t < a \\ I_0\{b\sqrt{(t^2-a^2)}\}, & \text{when } t > a \end{cases}$ |
| 34.† | $e^{-a\sqrt{(p^2-b^2)}}-e^{-ap}$ | $\begin{cases} 0, & \text{when } 0 < t < a \\ \dfrac{abI_1\{b\sqrt{(t^2-a^2)}\}}{\sqrt{(t^2-a^2)}}, & \text{when } t > a \end{cases}$ |

This table is not meant to be complete. But all the transforms used in this work are contained in it and a few additional ones, which arise in similar problems.

† In these $a$ is to be taken positive.

# INDEX

*The numbers refer to sections. Those of the Introduction are prefixed with I*

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